

# Sections, Homotopy Rational Points and Reductions of Curves

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**Abstract.** We study unramified sections of the fundamental group sequence of geometrically connected smooth projective curves of genus  $\geq 2$  over  $p$ -adic fields together with an integral model. We are particularly interested in the induced “specialized” sections of the special fibre and how they relate to homotopy rational points over the residue field. Under mild assumptions, such a specialized section induces a unique homotopy rational point of the special fibre that is compatible with the original section of the generic fibre in cohomological settings. We give two applications of such “specialized” homotopy rational points around the  $\ell$ -adic cycle class of a section.

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## Introduction

**Sections and homotopy rational points.** The profinite homotopy type of the spectrum of a field  $K$  is weakly equivalent to the classification space  $BG_K$  of the absolute Galois group of  $K$ . Thus, any  $K$ -variety  $X$  comes equipped with a morphism  $\widehat{\text{Et}}(X) \rightarrow \widehat{\text{Et}}(K) \simeq BG_K$  of profinite homotopy types and each  $K$ -rational point of  $X$  induces a splitting of this morphism. We call any such splitting  $BG_K \rightarrow \widehat{\text{Et}}(X)$  in the homotopy category of simplicial profinite sets over  $BG_K$  (cf. [Qui08]) a **homotopy rational point**. See Sect. 1 for a short summary of some basic facts on homotopy rational points.

Say,  $X$  is a  $K(\pi, 1)$ -space, i.e., the homotopy type of  $X$  is weakly equivalent to the classification space  $B\pi_1(X)$  of its fundamental group. This is the case e.g. for  $X/K$  any smooth curve except for rational projective curves (see [Sch96] Prop. 15). In this  $K(\pi, 1)$ -case, a homotopy rational point is equivalent (up to inner automorphisms) to a **section** of (the quotient map in) the exact sequence

$$\pi_1(X/K): \quad \mathbf{1} \longrightarrow \pi_1(X \otimes_K K^s, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow G_K \longrightarrow \mathbf{1} .$$

**Specialized sections and homotopy rational points.** Let us fix a  $p$ -adic field  $k$  with valuation ring  $\mathfrak{o}$  and residue field  $\mathbb{F}$ . Further, let  $X/k$  be a geometrically connected smooth projective curve of genus  $\geq 2$  together with a proper flat model  $\mathfrak{X}/\mathfrak{o}$  with reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}/\mathbb{F}$ . Let  $s$  be a section of the fundamental group sequence  $\pi_1(X/k)$ . We say that  $s$  **specializes** to a section  $\bar{s}$  of  $\pi_1(Y/\mathbb{F})$ , if  $s$  and  $\bar{s}$  are compatible via the specialization map of fundamental groups:

$$\begin{array}{ccc} G_k & \xrightarrow{s} & \pi_1(X) \\ \downarrow \text{can.} & & \downarrow \text{sp} \\ G_{\mathbb{F}} & \xrightarrow{\bar{s}} & \pi_1(Y) \end{array}$$

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(see [Sti12] Ch. 8). In fact, any section  $s$  specializes at least to a section  $\bar{s}_\ell$  of the geometrically pro- $\ell$  completed sequence  $\pi_1(Y/\mathbb{F})$  (this follows from [Sti12] Prop. 91 - see Lem. 3.5, below). Similarly, treating  $s$  as a homotopy rational point  $BG_k \rightarrow X$ , we say that  $s$  **specializes** to a homotopy rational point  $\bar{s}$  of  $Y/\mathbb{F}$ , if  $s$  and  $\bar{s}$  are compatible via the specialization morphism of homotopy types:

$$\begin{array}{ccc} BG_k & \xrightarrow{s} & \widehat{\text{Et}}(X) \\ \downarrow \text{can.} & & \downarrow \text{sp} \\ BG_{\mathbb{F}} & \xrightarrow{\bar{s}} & \widehat{\text{Et}}(Y) \end{array}$$

(see Def. 3.1, below). In case of bad reduction, it is no longer clear if the special fibre  $Y$  is still a  $K(\pi, 1)$ , so the immediate question arising is,

(†) *If a section  $s$  of  $\pi_1(X/k)$  specializes to a section of a sufficiently nice (but not necessarily  $K(\pi, 1)$ -) reduction  $Y/\mathbb{F}$ , does such a specialized section induce a specialized homotopy rational point of  $Y/\mathbb{F}$  ?*

Say, the section  $s$  of the fundamental group sequence  $\pi_1(X/k)$  specializes to a section  $\bar{s}$  of  $\pi_1(Y/\mathbb{F})$ . Suppose all the points in the normalization  $\pi : \tilde{Y} \rightarrow Y$  lying above singular points of  $Y$  contained in rational components are  $\mathbb{F}$ -rational. Studying the homotopy type of  $Y/\mathbb{F}$  (see Thm. 2.4, below), at least, a possible candidate for a specialized homotopy rational point of  $s$  giving back the specialized section  $\bar{s}$  does exist (see Cor. 3.3). However, the compatibility of  $s$  and this candidate for a specialized homotopy rational point (in abuse of notation also denoted by)  $\bar{s}$  via the specialization morphism of homotopy types is not clear. We will prove this compatibility at least in the sufficiently additive setting of cohomology cochains: Let  $\Lambda$  be a discrete torsion  $G_{\mathbb{F}}$ -module. The homotopy rational point  $s$  resp. the candidate  $\bar{s}$  induces a spitting  $s^*$  resp.  $\bar{s}^*$  of the canonical morphism  $\mathbb{R}\Gamma(k_{\text{ét}}, \Lambda) \rightarrow \mathbb{R}\Gamma(X_{\text{ét}}, \Lambda)$  resp.  $\mathbb{R}\Gamma(\mathbb{F}_{\text{ét}}, \Lambda) \rightarrow \mathbb{R}\Gamma(Y_{\text{ét}}, \Lambda)$  in the derived category. It turns out that these splittings are compatible via the specialization morphism of cohomology cochains (for the exact statement, see Thm. 4.1, below):

**Theorem A.** *Let  $X/k$  be a geometrically connected smooth projective curve of genus  $\geq 2$  over a  $p$ -adic field  $k$  together with a regular, proper, flat model  $\mathfrak{X}/\mathfrak{o}$  with reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}/\mathbb{F}$ . Suppose all the points in the normalization  $\pi : \tilde{Y} \rightarrow Y$  lying above singular points on rational components of  $Y$  are  $\mathbb{F}$ -rational. Let  $s$  be a section of  $\pi_1(X/k)$  specializing to a section of  $\pi_1(Y/\mathbb{F})$  and let  $\Lambda$  be a constructible  $G_{\mathbb{F}}$ -module. Then there is a homotopy rational point  $\bar{s}$  of  $Y/\mathbb{F}$  inducing the specialized section on fundamental groups and a commutative diagram of cohomology cochains in the derived category  $\mathcal{D}^+(\underline{\text{Ab}})$ :*

$$\begin{array}{ccc} \mathbb{R}\Gamma(X_{\text{ét}}, \Lambda) & \xrightarrow{s^*} & \mathbb{R}\Gamma(k_{\text{ét}}, \Lambda) \\ \uparrow \text{sp}^* & & \uparrow \\ \mathbb{R}\Gamma(Y_{\text{ét}}, \Lambda) & \xrightarrow{\bar{s}^*} & \mathbb{R}\Gamma(\mathbb{F}_{\text{ét}}, \Lambda). \end{array}$$

**Applications.** We will give two applications around the cycle class  $\text{cl}_s$  of a section (see [Sti12] Sect. 6.1, [EW09] or Rem. 4.10, below). The first is an application of specialized sections in general<sup>1</sup> and the second an application of Thm. A itself.

<sup>1</sup>In fact, it was the original motivation for question (†).

Let us shortly describe the first application: Based around a pro- $\ell$  specialization result, we give an independent proof to the following algebraicity result of Esnault and Wittenberg:

**Proposition B.** ([EW09] Cor. 3.4) *Let  $X/k$  be a geometrically connected smooth projective curve of genus  $\geq 2$  over a  $p$ -adic field  $k$ , admitting a section  $s$  of  $\pi_1(X/k)$ . Then the  $\ell$ -adic cycle class  $\text{cl}_s$  of  $s$  lies in the image of the Chern class map  $\hat{c}_1 : \text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow H^2(X, \mathbb{Z}_\ell(1))$  for each prime  $\ell \neq p$ .*

Using Tate-Lichtenbaum duality, this prime-to- $p$  algebraicity is equivalent to  $s^*$  trivializing the first Chern class map  $\hat{c}_1$  in  $H^2(G_k, \mathbb{Z}_\ell(1))$  for  $\ell \neq p$ . This is precisely the statement we will prove in Prop. 3.10, below. In fact, the pullback along  $s$  of certain Chern classes will turn out to be exactly the obstructions for the compatibility claimed in Thm. A. Thus, we have to avoid the usage of Thm. A in the proof of Prop. B resp. the equivalent Prop. 3.10 and modify  $X/k$  until it admits a  $K(\pi, 1)$ -model over  $\mathfrak{o}$ .

Our second application is a direct consequence of (a geometrically pro- $\ell$  completed variant of) Thm. A: In [EW09] Rem. 3.7 (iii) Esnault and Wittenberg raised the question whether the  $\ell$ -adic cycle class  $\text{cl}_s$  of a section  $s$  of  $\pi_1(X/k)$  admits a canonical lift to  $H^2(\mathfrak{X}, \mathbb{Z}_\ell(1))$ . Using the (unconditional) geometrically pro- $\ell$  completed specialization  $\bar{s}_\ell$  of  $s$ , we construct a canonical cycle class  $\text{cl}_s^\mathfrak{X}$  in  $H^2(\mathfrak{X}, \mathbb{Z}_\ell(1))$  for  $\mathfrak{X}/\mathfrak{o}$  a regular model satisfying the assumptions of Thm. A. A (geometrically pro- $\ell$  completed) variant of Thm. A shows that  $\text{cl}_s^\mathfrak{X}$  is indeed a lift of  $\text{cl}_s$  (see Prop. 4.12, below):

**Proposition C.** *Let  $X/k$  be a geometrically connected smooth projective curve of genus  $\geq 2$  over a  $p$ -adic field  $k$  together with a regular, proper, flat model  $\mathfrak{X}/\mathcal{O}$  satisfying the assumptions of Thm. A. Then for any  $\ell \neq p$  and any section  $s$  of  $\pi_1(X/k)$ , the induced  $\ell$ -adic cycle class  $\text{cl}_s$  admits a canonical lift  $\text{cl}_s^\mathfrak{X}$  to  $H^2(\mathfrak{X}, \mathbb{Z}_\ell(1))$ .*

**Notation.** In the following,  $k$  always denotes a  $p$ -adic field with valuation ring  $\mathfrak{o}$  and residue field  $\mathbb{F}$ . If  $K$  is any field, we denote a fixed separable closure by  $K^s$  and the corresponding absolute Galois group by  $G_K$ . Denote the maximal unramified subextension of  $k^s/k$  by  $k^{\text{nr}}/k$ . For  $X$  a  $K$ - resp.  $k$ -variety, denote its base-change to  $K^s$  resp.  $k^{\text{nr}}$  by  $X^s$  resp.  $X^{\text{nr}}$ . Denote by  $\hat{S}_{(*)}$  the category of (pointed) simplicial profinite sets together with the model structures of [Qui08]. For  $X$  a scheme together with a geometric point  $\bar{x}$ ,  $\pi_1(X, \bar{x})$  denotes its profinite étale fundamental group. Mostly we will skip the base-point in our notation. Denote by  $\widehat{\text{Et}}(X)$  its profinite étale homotopy type in  $\hat{S}_{(*)}$ . For a diagram  $\mathcal{Y}' \leftarrow \mathcal{Y} \rightarrow \mathcal{Y}''$  of simplicial (profinite) sets, we write  $\mathcal{Y}' \vee_{\mathcal{Y}} \mathcal{Y}''$  for the homotopy pushout. For a simplicial profinite set  $\mathcal{Y}$  with torsion local system  $\Lambda$ , write  $C^\bullet(\mathcal{Y}, \Lambda)$  for its cohomology cochains (see [Qui08] Sect. 2.2). If  $\mathcal{Y}$  is the homotopy type  $\widehat{\text{Et}}(X)$  of a scheme  $X$ , then  $C^\bullet(\widehat{\text{Et}}(X), \Lambda)$  is quasi-isomorphic to  $\mathbb{R}\Gamma(X_{\text{ét}}, \Lambda)$  (see [Qui08] Sect. 3.1). If  $\mathcal{Y}$  is the classification space  $BG$  of a profinite group  $G$ , then  $C^\bullet(BG, \Lambda)$  is quasi-isomorphic to  $\mathbb{R}\Gamma(G, \Lambda)$ . We will write just  $C^\bullet(X, \Lambda)$  resp.  $C^\bullet(G, \Lambda)$  in these cases. Similarly, we will just write  $C^\bullet(X \otimes_k k', \Lambda)$  for the  $G$ -equivariant cochains  $C^\bullet(\widehat{\text{Et}}(X) \times_{BG} EG, \Lambda)$  for  $K'/K$  a Galois extension with group  $G$  and  $X/K$  a  $K$ -variety. Finally, we will always use continuous étale cohomology in the sense of [Jan88].

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# 1 Preliminaries: Homotopy rational points

**Homotopical algebra.** We will work in the following homotopy categories:

**1.1** Let  $\hat{\mathcal{S}}$  be the category of simplicial profinite sets together with the model structures of [Qui08]. For  $G$  a profinite group, let  $BG$  be its profinite classification space and  $\hat{\mathcal{S}} \downarrow BG$  the category of simplicial profinite sets over  $BG$  together with the induced model structure. A simplicial profinite  $G$ -set is a simplicial profinite set together with a degreewise continuous  $G$ -action. Let  $\hat{\mathcal{S}}_G$  be the resulting category together with the model structure of [Qui10]. By [Qui10] Cor. 2.11,  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG)$  is Quillen equivalent to  $\mathcal{H}(\hat{\mathcal{S}}_G)$  via the base change functor  $\mathcal{X} \mapsto \mathcal{X} \times_{BG} EG$ , where  $BG$  is the profinite classification space of  $G$  and  $EG \rightarrow BG$  the universal covering. Under this equivalence, maps  $BG \rightarrow \mathcal{X}$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG)$  correspond to homotopy fixed points of  $\mathcal{X} \times_{BG} EG$ , i.e., maps  $\text{pt} \simeq EG \rightarrow \mathcal{X} \times_{BG} EG$  in  $\mathcal{H}(\hat{\mathcal{S}}_G)$ .

**1.2** Let  $\mathcal{X}$  be a simplicial profinite  $G$ -set. The set of homotopy fixed points  $[EG, \mathcal{X}]_{\hat{\mathcal{S}}_G}$  is the set of connected components of Quick's homotopy fixed point space  $\mathcal{X}^{hG} = S_G(EG, \mathcal{X}')$  (where  $\mathcal{X} \rightarrow \mathcal{X}'$  is a fibrant replacement in  $\hat{\mathcal{S}}_G$ ), defined and studied in [Qui10]. In general,  $\mathcal{X}^{hG}$  is difficult to describe. At least, by [Qui10] Thm. 2.16, there is a Bousfield-Kan type descent spectral sequence (with differentials in the usual “cohomological” directions)

$$E_2^{p,q} = H^p(G, \pi_{-q}(\mathcal{X})) \Rightarrow \pi_{-(p+q)}(\mathcal{X}^{hG}). \quad (1.1)$$

Applying Bousfield and Kan's connectivity lemma [BK72] Ch. IX 5.1 to the spectral sequence in 1.2, one can prove:

**1.3 Lemma.** *Let  $G$  be a profinite group of cohomological dimension  $\leq n$  and  $f : \mathcal{X} \rightarrow \mathcal{Y}$  an  $(n+1)$ -equivalence in  $\hat{\mathcal{S}}_G$  (i.e.,  $\pi_q(f)$  is an isomorphism for all  $q \leq n$  and an epimorphism for  $q = n+1$ ). Then  $f$  induces an injection*

$$[EG, \mathcal{X}]_{\hat{\mathcal{S}}_G} = \pi_0(\mathcal{X}^{hG}) \longrightarrow \pi_0(\mathcal{Y}^{hG}) = [EG, \mathcal{Y}]_{\hat{\mathcal{S}}_G}.$$

**Proof.** We may assume that  $\mathcal{Y}$  is fibrant and  $f$  is a fibration in  $\hat{\mathcal{S}}_G$ . Further, we may assume that  $\mathcal{X}^{hG}$  is non-empty. Say,  $s : E\Gamma \rightarrow \mathcal{X}$  is a model of a homotopy fixed point and let  $r$  be  $f \circ s$ . The fibre  $\mathcal{F}_s := \mathcal{X} \times_{\mathcal{Y}} EG$  comes equipped with a fibration into  $EG$ , hence is fibrant in  $\hat{\mathcal{S}}_G$ , too. Taking limits (i.e., forgetting the topology in [Qui10]) resp. simplicial mapping spaces  $S_\Gamma(EG, -)$  of  $\hat{\mathcal{S}}_G$  gives us a homotopy fibre sequence

$$\lim \mathcal{F}_s \longrightarrow \lim \mathcal{X} \longrightarrow \lim \mathcal{Y}$$

resp.

$$\mathcal{F}_s^{hG} \longrightarrow \mathcal{X}^{hG} \longrightarrow \mathcal{Y}^{hG}$$

in  $\underline{\mathbf{SSets}}_*$  (pointed by the neutral element in  $G$ ). By [Qui13] Lem. 2.9, the limit of  $f$  is an  $n$ -equivalence of simplicial sets. So, again by loc. cit., the first fibre sequence implies the  $n$ -connectedness of  $\mathcal{F}_s$ . Using the second homotopy fibre sequence, we get that the map of pointed sets  $(\pi_0(\mathcal{X}^{hG}), s) \rightarrow (\pi_0(\mathcal{Y}^{hG}), r)$  has kernel  $\pi_0(\mathcal{F}_s^{hG})$ . Bousfield and Kan's connectivity lemma applied to the descent spectral sequence (1.1) for  $\mathcal{F}_s$  implies that this kernel is trivial, since  $\mathcal{F}_s$  is  $n$ -connected and  $G$  has cohomological dimension  $\leq n$ . Varying over all the homotopy fixed points of  $\mathcal{X}$ , we get the result.  $\square$

**1.4** Let  $p : \pi \twoheadrightarrow G$  be an epimorphism of profinite groups with kernel  $\bar{\pi} \trianglelefteq \pi$ . By the adjunction between the profinite groupoid  $\Pi(-)$  and  $B(-)$ ,  $B\pi/BG$  is fibrant in  $\hat{\mathcal{S}} \downarrow BG$ . Thus,

$[BG, B\pi]_{\hat{\mathcal{S}} \downarrow BG}$  is given as  $\text{Hom}_{\hat{\mathcal{S}} \downarrow BG}(BG, B\pi)$  modulo homotopy equivalences over  $BG$  with respect to the standard cylinder object  $BG \otimes \Delta^1$ . Such homotopies between maps  $B(s_0)$  and  $B(s_1)$  for sections  $s_i$  of  $p$  correspond precisely to conjugation of these sections via elements of  $\bar{\pi}$ . In particular,  $B(-)$  resp.  $\pi_1(-, *)$  give canonical identifications between  $[BG, B\pi]_{\hat{\mathcal{S}} \downarrow BG}$  and the set of  $\bar{\pi}$ -conjugacy classes of section of  $p$ .

**1.5** Let  $\mathcal{X}/BG$  be a connected simplicial profinite set in  $\hat{\mathcal{S}} \downarrow BG$  and assume  $\pi_1(\mathcal{X}) \rightarrow G$  is an epimorphism. Then any map  $BG \rightarrow \mathcal{X}$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG)$  defines a  $\pi_1(\mathcal{X} \times_{BG} EG)$ -conjugacy class of splittings of the fundamental group sequence

$$1 \longrightarrow \pi_1(\mathcal{X} \times_{BG} EG) \longrightarrow \pi_1(\mathcal{X}) \longrightarrow G \longrightarrow 1$$

of  $\mathcal{X}/BG$ . Conversely, if the underlying simplicial profinite set  $\mathcal{X}$  of  $\mathcal{X}/BG$  is a  $K(\pi, 1)$  (i.e., the canonical map  $\mathcal{X} \rightarrow B\Pi(\mathcal{X})$  into the classification space of the profinite fundamental groupoid is a weak equivalence), it follows from 1.4 that sections of the above fundamental group sequence of  $\mathcal{X}/BG$  modulo conjugation correspond to maps  $BG \rightarrow \mathcal{X}$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG)$ .

Combining 1.5 with Lem. 1.3, we get:

**1.6 Corollary.** *Let  $G$  be a profinite group of cohomological dimension 1 and  $\mathcal{X}/BG$  a connected simplicial profinite set in  $\hat{\mathcal{S}} \downarrow BG$  s.t.  $\pi_1(\mathcal{X}) \rightarrow G$  is an epimorphism. Assume that the canonical map  $\mathcal{X} \times_{BG} EG \rightarrow B\Pi(\mathcal{X} \times_{BG} EG)$  admits a section in  $\mathcal{H}(\hat{\mathcal{S}}_G)$ . Then we get a canonical identification between the set of  $\pi_1(\mathcal{X} \times_{BG} EG)$ -conjugacy classes of sections of  $\pi_1(\mathcal{X}) \rightarrow G$  and  $[BG, \mathcal{X}]_{\hat{\mathcal{S}} \downarrow BG} \simeq [EG, \mathcal{X} \times_{BG} EG]_{\hat{\mathcal{S}}_G}$ .*

**Proof.** Indeed, the canonical map  $\mathcal{X} \times_{BG} EG \rightarrow B\Pi(\mathcal{X} \times_{BG} EG)$  is a 2-equivalence, so it induces an injection on the respective sets of homotopy fixed points by Lem. 1.3. Since it admits a section, the induced map on homotopy fixed points is even bijective and the claim follows from 1.5.  $\square$

**Homotopy rational points.** We are mainly interested in profinite homotopy types of varieties over a field  $K$ :

**1.7** Let  $Z$  be a  $K$ -variety. The profinite étale homotopy type of the spectrum of  $K$  (pointed by the choice of a separable closure  $K^s/K$ ) is a  $K(\pi, 1)$  with fundamental group  $G_K$ , i.e., it is weakly equivalent to the profinite classification space  $BG_K$  in the pointed category  $\hat{\mathcal{S}}_*$ . We define the profinite homotopy type  $\widehat{\text{Et}}(Z/K) \rightarrow BG_K$  of  $Z/K$  as the resulting map in  $\hat{\mathcal{S}} \downarrow BG_K$  induced by the structural map of  $Z/K$ . Using [Qui10] Thm. 3.5 and Lem. 3.3, we see that the underlying homotopy type of  $\widehat{\text{Et}}(Z/K) \times_{BG_K} EG_K$  together with its abstract  $G_K$ -action corresponds to the homotopy type  $\widehat{\text{Et}}(K \otimes_K K^s)$  together with the induced abstract  $G_K$ -action. Similar arguments work for any Galois extension  $K'/K$  and  $Z \otimes_K K'$ , too.

**1.8** Each  $K$ -rational point of  $Z$  defines a map  $BG_K \rightarrow \widehat{\text{Et}}(Z/K)$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_K)$ , i.e., a homotopy fixed point of  $\widehat{\text{Et}}(Z/K) \times_{BG_K} EG_K$ . Thus, for any simplicial profinite set  $\mathcal{X}/BG_K$ , we call any map  $BG_K \rightarrow \mathcal{X}$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_K)$  a **homotopy rational point** of  $\mathcal{X}$  over  $K$ . A homotopy rational point of  $Z/K$  simply is a homotopy rational point of  $\widehat{\text{Et}}(Z/K)$ .

**1.9** Let  $Z/K$  be a geometrically connected  $K$ -variety. Then any homotopy rational point of  $Z/K$  gives a conjugacy class of splittings of the fundamental group sequence  $\pi_1(Z/K)$ . Conversely, assume  $Z$  has the  $K(\pi, 1)$ -property, i.e., its étale cohomology of constructible locally constant coefficients is given by the cohomology of its finite étale site. It follows that  $\widehat{\text{Et}}(Z/K)$  is a  $K(\pi, 1)$ -space in the above sense. By 1.5, we get a canonical identification between the set

of  $\pi_1(Z \otimes_K K^s)$ -conjugacy classes of the fundamental group sequence  $\pi_1(X/K)$  and homotopy rational resp. homotopy fixed points of  $Z/K$  (cf. [Qui10] Sect. 3.2). By [Sti02] Prop. A.4.1, this in particular is the case for  $Z$  any smooth curve over  $K$  except for Brauer-Severi curves.

**1.10 Lemma.** *Let  $Z$  be a non-empty irreducible  $K$ -variety. Then the canonical map*

$$Z(K) \longrightarrow [B\Gamma_K, \widehat{\text{Et}}(Z/K)]_{\hat{S} \downarrow B\Gamma_K}$$

*is trivial in the following two situations:*

- (i) *If  $K$  has cohomological dimension  $\leq n$  and  $\widehat{\text{Et}}(Z/K) \times_{BG_K} EG_K$  is  $n$ -connected.*
- (ii) *If  $K$  has characteristic 0 and  $Z$  is  $\mathbb{A}^1$ -chain connected (i.e., for any two  $K$ -rational points  $z', z''$ , there are finitely many  $K$ -morphisms  $u_i : \mathbb{A}_K^1 \rightarrow Z$ ,  $1 \leq i \leq n$  with  $u_1(0) = z'$ ,  $u_n(1) = z''$  and  $u_i(1) = u_{i+1}(0)$ ).*

**Proof.** To prove (i), it is enough to show that  $\pi_0((\widehat{\text{Et}}(Z/K) \times_{BG_K} E\Gamma_K)^{h\Gamma_K})$  is trivial. But this follows from Bousfield and Kan's connectivity lemma applied to the descent spectral sequence (1.1) for  $\widehat{\text{Et}}(Z/K) \times_{BG_K} E\Gamma_K$ :  $\text{cd}(K) \leq n$  and  $\widehat{\text{Et}}(Z/K) \times_{BG_K} EG_K$  is  $n$ -connected by assumption. Statement (ii) holds, since  $\mathbb{A}_K^1$  is contractible (to  $B\Gamma_K$ ) in characteristic 0.  $\square$

**Geometric pro- $\ell$  completions.** Let us shortly discuss pro- $\ell$ -completion in  $\mathcal{H}(\hat{S}_G)$  for a strongly complete profinite group  $G$ .

**1.11** In [Qui12], Quick gave an explicit construction of a pro-finite completion in  $\hat{S}_G$ . An analogue construction gives a pro- $\ell$ -completion in  $\hat{S}_G$ , too (see loc. cit. Rem. 3.3). Let us shortly describe this construction: By loc. cit. 4.3, any profinite  $G$ -space  $\mathcal{X}$  is isomorphic to a profinite  $G$ -space of the form  $\{\mathcal{X}_i\}_{i \in I}$  for  $\mathcal{X}_i$  a finite discrete  $G$ -space. Then the pro- $\ell$ -completion is given as the profinite  $G$ -space

$$\mathcal{X}_\ell^\wedge := \{\bar{W}\hat{\Gamma}_\ell(\mathcal{X}_i)\}_{i \in I},$$

where  $\bar{W}(-)$  is (levelwise) the classification space and  $\hat{\Gamma}_\ell(\mathcal{X}_i)$  is degreeewise the pro- $\ell$  completion of the free loop group  $\Gamma(\mathcal{X}_i)$  of  $\mathcal{X}_i$ . Arguing directly using the (levelwise) homotopy fibre sequence

$$\hat{\Gamma}_\ell(\mathcal{X}_i) \longrightarrow W\hat{\Gamma}_\ell(\mathcal{X}_i) \longrightarrow \bar{W}\hat{\Gamma}_\ell(\mathcal{X}_i),$$

or comparing  $\mathcal{X}_\ell^\wedge$  with the fibrant replacement in Morel's pro- $\ell$  model structure in [Mor96] (see Sect. 2.1 in loc. cit.), we get that  $\pi_1(\mathcal{X}_\ell^\wedge)$  equals the pro- $\ell$  completion  $\pi_1^\ell(\mathcal{X})$ ,  $\mathcal{X}_\ell^\wedge$  has pro- $\ell$  profinite homotopy groups and the canonical map  $\mathcal{X} \rightarrow \mathcal{X}_\ell^\wedge$  induces an isomorphism in  $\mathcal{D}^+(\underline{\text{Mod}}_G)$  on cohomology cochains  $C^\bullet(-, \Lambda)$  for any finite  $\ell$ -torsion  $G$ -module  $\Lambda$ .

For  $\mathcal{X}/BG$  in  $\hat{S} \downarrow BG$ , denote by  $\mathcal{X}_\ell^\wedge/BG$  the homotopy type in  $\mathcal{H}(\hat{S} \downarrow BG)$  corresponding to the pro- $\ell$  completion  $(\mathcal{X} \times_{BG} EG)_\ell^\wedge$  in  $\mathcal{H}(\hat{S}_G)$ .

If  $G$  itself is not a pro- $\ell$  group,  $\mathcal{X}_\ell^\wedge/BG$  corresponds to a “geometric” pro- $\ell$  completion in the relative homotopy category  $\mathcal{H}(\hat{S} \downarrow BG)$ . Let us discuss the case of  $B\pi \rightarrow BG$  for suitable  $\pi \twoheadrightarrow G$ :

**1.12** Let  $p : \pi \twoheadrightarrow G$  be an epimorphism of profinite groups with kernel  $\bar{\pi} \trianglelefteq \pi$ . Assume  $\bar{\pi}$  is an  $\ell$ -good profinite group, i.e., the pro- $\ell$  completion map  $\bar{\pi} \rightarrow \hat{\pi}_\ell$  induces isomorphisms  $H^q(\bar{\pi}, \Lambda) \simeq H^q(\hat{\pi}_\ell, \Lambda)$  for all finite  $\ell$ -torsion  $\hat{\pi}_\ell$ -modules  $\Lambda$  and all these cohomology groups are finite. Let  $\Delta_\ell \leq_c \bar{\pi}$  be the kernel of the pro- $\ell$  completion of  $\bar{\pi}$ . Note that it is a characteristic subgroup by the universal property of the completion. Then we define the geometric pro- $\ell$



completion of  $\pi \rightarrow G$  as  $\hat{\pi}_{(\ell)} := \pi/\Delta_\ell \rightarrow G$ . By construction, the geometric pro- $\ell$  completion  $\pi \rightarrow \hat{\pi}_{(\ell)}$  sits in the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbf{1} & \longrightarrow & \bar{\pi} & \longrightarrow & \pi & \longrightarrow & G \longrightarrow \mathbf{1} \\ & & \downarrow & & \downarrow & & \parallel \\ \mathbf{1} & \longrightarrow & \hat{\pi}_\ell & \longrightarrow & \hat{\pi}_{(\ell)} & \longrightarrow & G \longrightarrow \mathbf{1}. \end{array}$$

It follows that  $B\hat{\pi}_{(\ell)} \times_{BG} EG \rightarrow (B\hat{\pi}_{(\ell)} \times_{BG} EG)_\ell^\wedge$  is a weak equivalence in  $\hat{\mathcal{S}}_G$  (this is a special case of the pro- $\ell$  analogue of [Qui12] Thm. 3.14). Further, the canonical Map  $\pi \rightarrow \hat{\pi}_{(\ell)}$  induces an isomorphism  $(B\pi \times_{BG} EG)_\ell^\wedge \rightarrow B\hat{\pi}_{(\ell)} \times_{BG} EG$ . In particular,  $(B\pi)_\ell^\wedge = B(\hat{\pi}_{(\ell)})$ .

For fundamental groups of geometrically connected  $K$ -varieties we define:

**1.13** Let  $Z/K$  be a geometrically connected  $K$ -variety. Then we write  $\pi_1^\ell(Z)$  resp.  $\pi_1^{(\ell)}(Z)$  for the pro- $\ell$  resp. geometric pro- $\ell$  completion of the étale fundamental group  $\pi_1(Z)$  in the sense of 1.12. Denote by  $\pi_1^{(\ell)}(Z/\mathbb{F})$  the geometrically pro- $\ell$  completed fundamental group sequence

$$\mathbf{1} \longrightarrow \pi_1^\ell(Z \otimes_K K^s) \longrightarrow \pi_1^{(\ell)}(Z) \longrightarrow G_K \longrightarrow \mathbf{1}.$$

Say  $G_K$  is strongly complete (e.g.,  $K$  a finite or  $p$ -adic field). Then  $\pi_1^{(\ell)}(Z)$  is the fundamental group of the geometric pro- $\ell$  completion  $\widehat{\text{Et}}(Z/K)_{(\ell)}^\wedge$ .

## 2 The étale homotopy type of a curve

**The étale homotopy type of a curve** We want to discuss the the étale homotopy type of a curve  $Z$  over a field  $K$ .

**2.1** Recall that for  $Z$  a reduced  $K$ -variety, its semi- resp. weak-normalization  $\pi^{\text{sn}} : Z^{\text{sn}} \rightarrow Z$  resp.  $\pi^{\text{wn}} : Z^{\text{wn}} \rightarrow Z$  is universal among factorizations  $f : Z' \rightarrow Z$  of the normalization  $\pi : \tilde{Z} \rightarrow Z$  with  $f$  birational, bijective on points and inducing trivial resp. purely inseparable extensions on the residue fields of all points (cf. [Kol96] Sect. I.7.2). In particular, for  $Z$  a curve over a perfect field, the semi- and weak-normalization coincide. Further, the curve  $Z$  itself is semi-normal (i.e., agrees with its semi-normalization), if and only if geometrically it has at most ordinary multiple points as singularities (use [Kol96] Sect. I 7.2.2.1).

**2.2** Let  $Z$  be a reduced curve over  $K$  and assume either  $Z$  to be semi-normal or  $K$  to be perfect. Denote by  $\Gamma(Z)$  the following bipartite graph: The two sets of nodes are the irreducible components resp. the singular points of  $Z^{\text{sn}}$  and a node corresponding to a component  $Z_i$  is joined to a node corresponding to a singularity  $z$  by multiplicity-of- $z$ -in- $Z_i$ -many edges. Let  $\Gamma.(Z)$  in  $\hat{\mathcal{S}}$  be the profinite completion of the canonical realization of  $\Gamma(Z)$  as a simplicial set. We will call both  $\Gamma(Z)$  and  $\Gamma.(Z)$  the **dual-(bipartite)graph** of  $Z$ . Obviously,  $\Gamma.(Z)$  has the homotopy type of the profinite completion of a bouquet of  $S^1$ 's, i.e., is isomorphic to  $BF_r$  for  $F_r$  the free profinite group of  $r$  generators and  $r$  the number of loops in  $\Gamma(Z)$ . If  $Z$  is even semi-stable, then  $\Gamma(Z)$  is just the barycentric subdivision of the regular dual-graph of the semi-stable curve  $Z$ . It follows from the universal property of weak-normalization (see [AB69] Thm. 4) that weak-normalization, hence the dual-graph  $\Gamma.(Z)$ , is functorial with respect to non-constant finite morphisms.

**2.3** Suppose moreover all points in the normalization  $\tilde{Z}$  lying above a singular point  $z$  of  $Z$  are  $k(z)$ -rational. Denote by  $\Sigma.(z)$  the star in  $\Gamma.(Z)$  of a node corresponding to the singular

point  $z$  and by  $\mathcal{Z}^*$  the étale homotopy type of the normalization  $\tilde{Z}$  glued to  $\coprod_z BG_{k(z)} \otimes \Sigma.(z)$  via the points above the singular points  $z$  of  $Z$ . Contracting all the stars  $\Sigma.(z)$ , we get a canonical factorization of the homotopy type of the normalization  $\pi$  over the canonical morphism  $\mathcal{Z}^* \rightarrow \widehat{\text{Et}}(Z/K)$ , functorial with respect to non-constant finite morphisms.

**2.4 Theorem.** *Let  $Z$  be a reduced, connected curve over  $K$  and assume either  $Z$  to be semi-normal or  $K$  to be perfect. Denote by  $\pi_i : \tilde{Z}_i \rightarrow Z_i$  the normalizations of the irreducible components and let  $\mathcal{R} = \mathcal{R}(Z)$  be the set of indices  $i$  s.t.  $\tilde{Z}_i$  is a rational projective component.*

- (i) *Suppose all points in the normalization  $\tilde{Z}$  lying above a singular point  $z$  of  $Z$  are  $k(z)$ -rational. Then  $\mathcal{Z}^* \rightarrow \widehat{\text{Et}}(Z/K)$  is a weak equivalence in  $\hat{\mathcal{S}} \downarrow BG_K$  (cf. 2.3).*
- (ii) *If  $K$  is separably closed, then*

$$\widehat{\text{Et}}(Z/K) \simeq \left( \bigvee_i \widehat{\text{Et}}(\tilde{Z}_i/K) \right) \vee \Gamma.(Z)$$

*holds (functorially in non-constant finite morphisms) in the homotopy category  $\mathcal{H}(\hat{\mathcal{S}})$ .*

- (iii) *Suppose  $K$  has characteristic 0 or cohomological dimension  $\leq 1$  and all points in the normalization  $\pi : \tilde{Z} \rightarrow Z$  lying above singular points of  $Z$  contained in rational projective components are  $K$ -rational. Then for each rational component  $\tilde{Z}_i$  there is a section  $s_i$  of  $\pi_1(Z/K)$  s.t.*

$$\widehat{\text{Et}}(Z/K) \simeq (B\pi_1(Z)) \vee_{\coprod_i s_i, \coprod_i x_i} \left( \prod_{i \in \mathcal{R}} \widehat{\text{Et}}(\tilde{Z}_i/K) \right)$$

*holds in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_K)$  for  $x_i$  an arbitrary  $K$ -rational point of  $\tilde{Z}_i$ .*

**2.5 Remark.** If one works with pro-simplicial sets together with Isaksen's model structure (see [Isa01]), the proof below for Thm. 2.4 (i), for (ii) and for the characteristic 0 case of (iii) still goes through without profinite completion. For the remaining case of (iii) an analogue of Lem. 1.10 (i) is needed, e.g. an adequate analogue of the Quillen equivalence  $\mathcal{H}(\hat{\mathcal{S}} \downarrow B\Gamma) \simeq \mathcal{H}(\hat{\mathcal{S}}_\Gamma)$  in [Qui10] 2.11 and for the descent spectral sequence of loc. cit. Thm. 2.16.

**2.6 Remark.** Let  $f : Z' \rightarrow Z$  be a finite non-constant morphism of curves as in Thm. 2.4 (iii). Suppose that there is no non-rational projective component  $\tilde{Z}'_j$  of  $Z'$  lying over a rational projective component  $\tilde{Z}_i$  of  $Z$ . Then the proof of Thm. 2.4 will show that  $f$  is compatible with

$$(B\pi_1(f)) \vee_{\coprod_i s'_i, \coprod_j x'_j} \left( \prod_{j \in \mathcal{R}(Z')} f|_{\tilde{Z}'_j} \right)$$

for compatible choices of the  $x_i$  and  $s_i$ . If there is a non-rational component  $\tilde{Z}'_j$  over a rational projective one  $\tilde{Z}_i$ , then this is no longer true: E.g.  $H^2(\tilde{Z}_i) \rightarrow H^2(\tilde{Z}'_j)$  is invisible for  $B\pi_1(f)$ .

**2.7 Remark.** For  $S$  a  $Z$ -scheme and  $z$  a closed point of  $Z$  we just write  $S_z^h$  resp.  $S^\bullet$  for the henselization  $S \times_Z \text{Spec}(\mathcal{O}_{Z,z}^h)$  resp. for the punctured scheme  $S \times_Z Z \setminus \{z\}$ . Before giving the proof of Thm. 2.4, let us first recall that the (punctured) étale tubular neighbourhood satisfies

$$T_{Z,z}^{(\bullet)} \simeq \widehat{\text{Et}}(Z_z^{h,(\bullet)}) :$$

We work with Cox's model for the (punctured) tubular neighbourhood in  $\text{Pro}\mathcal{H}(\underline{\text{SSets}}_\bullet)$  (see [Cox78]). The non-punctured case is covered by loc. cit. Thm. 2.2. For the punctured case, use

$$T_{Z,z}^\bullet = \lim_{\mathfrak{V} \in t_{Z,z}^{\text{Ho}}} \widehat{\text{Et}}(\mathfrak{V}^\bullet) = \lim_{(U,u)} \lim_{\mathfrak{V} \in t_{U,u}^{\text{Ho}}} \widehat{\text{Et}}(\mathfrak{V}^\bullet),$$



where  $(U, u)$  runs through the strict étale neighbourhoods of  $z$  and  $t_{U,u}^{\text{Ho}}$  is the opposite category of the homotopy category of the full subcategory  $t_{U,u}$  of degreewise Noetherian and separated simplicial objects  $\mathfrak{V}$ . in  $U_{\text{ét}}$  s.t.  $u^*\mathfrak{V}$ . is a hypercover of  $k(u)$  ( $= k(z)$ ). We have to compare  $\pi_0$ ,  $\pi_1$  and cohomology of the respective homotopy types. Arguing degreewise and using the descriptions of these homotopy invariants in [Fri82] Ch. 5 (together with the spectral sequence in loc. cit. Prop. 2.4), we may restrict to simplicial schemes of the form  $\mathfrak{V} = \text{cosk}_m^U \mathfrak{V}$ . in  $t_{U,u}$  for  $m \gg 0$ . After a possible refinement of  $(U, u)$ , such an object in  $t_{U,u}$  can be refined to a simplicial object  $\mathfrak{V}$ . in the finite étale site  $U_{\text{ét}}$ . It follows that the canonical morphisms  $\mathfrak{V}_n \rightarrow (\text{cosk}_{n-1}^U \mathfrak{V})_n$  are finite étale. Since  $u^*\mathfrak{V}$ . is a hypercover of  $k(u)$ , these maps are surjective on connected components, i.e., surjective and  $\mathfrak{V}$ . is a hypercover of  $U$ . In particular,  $\mathfrak{V}^\bullet \rightarrow U^\bullet$  is a hypercover and hence a weak equivalence. Summing up, we get

$$\lim_{(U,u)} \lim_{\mathfrak{V} \in t_{U,u}^{\text{Ho}}} \widehat{\text{Et}}(\mathfrak{V}^\bullet) \simeq \lim_{(U,u)} \widehat{\text{Et}}(U^\bullet) \simeq \widehat{\text{Et}}(Z_z^{h,\bullet})$$

and hence the claim follows.

**Proof of Thm. 2.4.** Weak-normalization is a universal homeomorphism by [AB69] Thm. 4, i.e., we may assume that  $Z$  is semi-normal in the perfect case, as well. Factor the normalization into finitely many steps

$$\tilde{Z} = Z^{(n)} \xrightarrow{\pi^{(n-1)}} Z^{(n-1)} \xrightarrow{\pi^{(n-2)}} \dots \xrightarrow{\pi^{(0)}} Z^{(0)} = Z,$$

where in each step we resolve exactly one of the ordinary multiple points. Say in the  $(l+1)^{\text{th}}$  step  $\pi^{(l)} : Z^{(l+1)} \rightarrow Z^{(l)}$  the multiple point  $z^{(l)}$  and let  $z_1^{(l)}, \dots, z_{r_l}^{(l)}$  be the  $r_l$  distinct points of  $Z^{(l+1)}$  over  $z^{(l)}$ . We get the homotopy pushout

$$\widehat{\text{Et}}(Z^{(l+1)}/K) \vee_{z_1^{(l)}, \dots, z_{r_l}^{(l)}} (BG_{k(z^{(l)})} \otimes \Sigma.(z^{(l)}))$$

in  $\hat{\mathcal{S}} \downarrow BG_K$  by joining  $z_1^{(l)}, \dots, z_{r_l}^{(l)}$  via the ends of the rays of the star  $BG_{k(z^{(l)})} \otimes \Sigma.(z^{(l)})$ . By contracting  $BG_{k(z^{(l)})} \otimes \Sigma.(z^{(l)})$  to  $z^{(l)}$ ,  $\pi^{(l)}$  induces a canonical morphism

$$\bar{\pi}^{(l)} : \widehat{\text{Et}}(Z^{(l+1)}/K) \vee_{z_1^{(l)}, \dots, z_{r_l}^{(l)}} (BG_{k(z^{(l)})} \otimes \Sigma.(z^{(l)})) \longrightarrow \widehat{\text{Et}}(Z^{(l)}/K).$$

(i): It suffices to see that  $\bar{\pi}^{(l)}$  is a weak equivalence in  $\hat{\mathcal{S}} \downarrow BG_K$ . Clearly, it is enough to deal only with the first step ( $l = 0$ ). To ease the notation, we skip the upper subscripts and write just  $Z'$  for  $Z^{(1)}$ ,  $z$  for  $z^{(0)}$  and  $z_1, \dots, z_r$  for  $z_1^{(0)}, \dots, z_{r_0}^{(0)}$ . By [Fri82] Prop. 15.6, the canonical morphism  $\widehat{\text{Et}}(Z^\bullet/K) \vee_{T_{Z,z}^\bullet} T_{Z,z} \rightarrow \widehat{\text{Et}}(Z/K)$  is a weak equivalence ( $Z^\bullet$  is no longer connected, but the proof goes through without any changes), where the (punctured) tubular neighbourhood  $T_{Z,z}^{(\bullet)}$  is weakly equivalent to the étale homotopy type of the (punctured) henselization  $Z_z^{h,(\bullet)}$  by Rem. 2.7 (and similar for  $Z'$ ). By prime avoidance,  $Z_z^{h,\bullet}$  is affine of dimension 0 and hence discrete, where the distinct points are given by the function fields of the components  $Z_i$  containing  $z$ . Summing up, we have

$$Z^\bullet = Z'^\bullet = Z' \setminus \{z_0, \dots, z_r\},$$

$$T_{Z,z} \simeq BG_{k(z)} \simeq T_{Z',z_i},$$

$$T_{Z,z}^\bullet \simeq \coprod_i T_{Z',z_i}^\bullet,$$

$$\widehat{\text{Et}}(Z'/K) \simeq (\dots (\widehat{\text{Et}}(Z'^\bullet/K) \vee_{T_{Z',z_1}^\bullet} T_{Z',z_1}) \vee_{T_{Z',z_2}^\bullet} \dots) \vee_{T_{Z',z_{i-1}}^\bullet} T_{Z',z_{i-1}},$$

where the last statement is again [Fri82] Prop. 15.6 applied multiple times. Here we used that all points  $z_i$  are  $k(z)$ -rational by assumption. We get  $\widehat{\text{Et}}(Z'/K) \simeq \widehat{\text{Et}}(Z'^{\bullet}/K) \vee_{(\coprod_i T_{Z',z_i}^{\bullet})} (\coprod_i T_{Z',z_i})$ , hence

$$\begin{aligned} & \widehat{\text{Et}}(Z'/K) \vee_{z_0, \dots, z_r} (BG_{k(z)} \otimes \Sigma.(z)) \\ & \simeq (\widehat{\text{Et}}(Z'^{\bullet}/K) \vee_{(\coprod_i T_{Z',z_i}^{\bullet})} (\coprod_i T_{Z',z_i})) \vee_{(\coprod_i T_{Z',z_i})} (T_{Z,z} \otimes \Sigma.(z)) \\ & \simeq \widehat{\text{Et}}(Z^{\bullet}/K) \vee_{T_{Z,z}^{\bullet}} T_{Z,z}, \end{aligned}$$

i.e.  $\bar{\pi}^{(0)}$  is a weak equivalence, just as claimed. If we put all the single steps  $\bar{\pi}^{(l)}$  together, we get a weak equivalence in  $\hat{\mathcal{S}} \downarrow BG_K$  between the étale homotopy types of  $Z$  and of its normalization  $\tilde{Z}$  with  $z_0^{(l)}, \dots, z_{r_k}^{(l)}$  joined together as the ends of rays of  $\Sigma.(z_l)$ , i.e., between  $\widehat{\text{Et}}(Z/K)$  and  $\mathcal{Z}^*$ .

(ii): If  $K$  is separably closed, the assumptions of (ii) clearly are satisfied. Further, by Lem. 1.10 (i), we can move all the  $z_j^{(k)}$ 's on one component  $\tilde{Z}_i$  to a single  $K$ -point  $x_i$  of  $\tilde{Z}_i$ . As a result,  $\widehat{\text{Et}}(Z/K)$  is weakly equivalent to the dual graph  $\Gamma.(Z)$  with all the  $\widehat{\text{Et}}(\tilde{Z}_i/K)$  glued via  $x_i$  to the corresponding node of  $\Gamma.(Z)$ . Since  $\Gamma.(Z)$  is connected, we get the desired equivalence between  $\widehat{\text{Et}}(Z/K)$  and  $(\bigvee_i \widehat{\text{Et}}(\tilde{Z}_i/K)) \vee \Gamma.(Z)$ .

(iii): First, note we may assume that  $Z$  itself is non-rational. Since  $Z$  is connected, each rational projective component  $\tilde{Z}_i$  contains a rational point, i.e., is isomorphic to  $\mathbb{P}_K^1$ .

Let  $Z' \hookrightarrow Z$  be the closed subscheme given by the union of components  $Z_i$  for  $i \notin \mathcal{R}$  and let  $Z'' \rightarrow Z'$  be the normalization at all singular points of  $Z$  contained in rational projective components. By assumption, all points of  $Z''$  over such singular points are  $K$ -rational. The cohomology resp. group-cohomology in degrees  $\geq 1$  of one point unions resp. free products is the direct sum of the cohomology of the factors. As a bouquet of  $S^1$ 's,  $\Gamma.(Z'')$  is a  $K(\pi, 1)$ . Since the  $\tilde{Z}_i$  are  $K(\pi, 1)$  for every  $i \notin \mathcal{R}$ , too (see [Sti02] Prop. A.4.1),  $Z''$  is a  $K(\pi, 1)$ . Hence the same is true for  $Z''$  itself.

Arguing as in (i), we get that  $\widehat{\text{Et}}(Z/K)$  is weakly equivalent to the étale homotopy type of  $Z'' \amalg \coprod_{i \in \mathcal{R}} \tilde{Z}_i$  glued together via the stars  $BG_K \otimes \Sigma.(z)$  corresponding to singular points  $z$  in  $Z$  contained in rational projective components (cf. the construction of  $\mathcal{Z}^*$  in 2.3). Denote the latter homotopy type by  $\mathcal{Z}_{\mathcal{R}}^*$ , i.e., we have a weak equivalence  $\mathcal{Z}_{\mathcal{R}}^* \rightarrow \widehat{\text{Et}}(Z/K)$ . Let  $\mathcal{Z}_{K(\pi,1)}$  be the homotopy type we get from  $\mathcal{Z}_{\mathcal{R}}^*$  by contracting the étale homotopy types of the rational projective components  $\tilde{Z}_i$  to  $BG_K$  for each  $i \in \mathcal{R}$  individually. It follows from the above discussion, that  $\mathcal{Z}_{K(\pi,1)}$  is a  $K(\pi, 1)$ -space, weakly equivalent to  $B\pi_1(Z)$ .

By Lem. 1.10 we can move all the  $z_j^{(l)}$ 's on one rational projective component  $\tilde{Z}_i$  to a single  $K$ -point  $x_i$  in  $\hat{\mathcal{S}} \downarrow BG_K$ . It follows that  $\mathcal{Z}_{\mathcal{R}}^*$  and  $\mathcal{Z}_{K(\pi,1)} \vee_{\coprod_i BG_K} (\coprod_{i \in \mathcal{R}} \widehat{\text{Et}}(\tilde{Z}_i/K))$  are weakly equivalent via the  $K$ -points  $x_i$  and gluing-morphisms  $BG_K \rightarrow \mathcal{Z}_{K(\pi,1)}$  induced by the contracted components in  $\mathcal{Z}_{K(\pi,1)}$ . This translates to our claim, if we let  $s_i$  be the sections of  $\pi_1(Z/K)$  corresponding to these gluing morphisms under the weak equivalence  $\mathcal{Z}_{K(\pi,1)} \simeq B\pi_1(Z)$ .  $\square$

Let us formulate some immediate consequences of Thm. 2.4:

**2.8 Corollary.** *Let  $Z/K$  be a curve as in Thm. 2.4.*

- (i)  $Z$  is a  $K(\pi, 1)$  if and only if all the  $\tilde{Z}_i$ 's are  $K(\pi, 1)$ , i.e., if none of the  $\tilde{Z}_i$ 's is a Brauer-Severi curve.
- (ii) If  $K$  is separably closed, then

$$\pi_1(Z) \simeq (*_{i \notin \mathcal{R}} \pi_1(\tilde{Z}_i)) * F_r.$$

In particular, for any prime  $\ell$ ,  $\pi_1(Z)$  is  $\ell$ -good if and only if all the  $\pi_1(\tilde{Z}_i)$ 's are  $\ell$ -good.

(iii) Suppose  $Z/K$  satisfies the assumption of Thm. 2.4 (iii). Then for each constructible local system  $\Lambda$  on  $Z$ , we get an exact triangle of cohomology cochains

$$\begin{array}{ccc} C^\bullet(Z, \Lambda) & \longrightarrow & C^\bullet(\pi_1(Z), \Lambda) \oplus \bigoplus_{i \in \mathcal{R}} C^\bullet(\tilde{Z}_i, \Lambda) \\ & \nwarrow +1 & \downarrow \oplus_i (s_i^* + x_i^*) \\ & & \bigoplus_{i \in \mathcal{R}} C^\bullet(G_K, \Lambda) \end{array}$$

(iv) Moreover, suppose  $K$  has cohomological dimension  $\leq 1$ . Then the higher cohomology groups ( $q = 2, 3$ ) are given by

$$H^q(Z, \Lambda) = H^q(\pi_1(Z), \Lambda) \oplus \bigoplus_{i \in \mathcal{R}} H^q(\tilde{Z}_i, \Lambda).$$

**2.9 Remark.** Using [Sti06] Ex. 5.5, we get the statement corresponding to (iii) for the étale fundamental group of  $Z$  a semi-stable curve.

**Proof of Cor. 2.8.** To prove (i) it suffices to see that  $Z \otimes_K K^s$  is a  $K(\pi, 1)$ , i.e., we may assume that  $K$  is separably closed. The cohomology resp. group-cohomology in degrees  $\geq 1$  of one point unions resp. free products is the direct sum of the cohomology of the factors. As a bouquet of  $S^1$ 's,  $\Gamma(Z)$  is a  $K(\pi, 1)$ . Thus, (i) follows from Thm. 2.4 (ii).

The first statement of (ii) is again a direct consequence of Thm. 2.4 (ii). For the second statement we argue as in (i) using that pro- $\ell$  completion preserves free products.

For  $\mathcal{Z}_{K(\pi, 1)} \simeq B\pi_1(Z)$  as in the proof of Thm. 2.4 (iii), loc. cit. gives the homotopy cofibre sequence

$$\coprod_{i \in \mathcal{R}} BG_K \xrightarrow{(\coprod_i s_i) \sqcup (\coprod_i x_i)} \mathcal{Z}_{K(\pi, 1)} \sqcup (\coprod_{i \in \mathcal{R}} \widehat{\text{Et}}(\tilde{Z}_i/K)) \xrightarrow{\text{can}} Z$$

inducing the exact triangle of cohomology cochains, claimed in (iii).

Finally, (iv) is a direct consequence of the long exact sequence of cohomology groups induced by the triangle in (iii).  $\square$

**2.10 Corollary.** Let  $Z/K$  be a curve as in Thm. 2.4 for  $K$  separably closed and  $\ell \neq \text{char}(K)$ . Then  $\pi_1(Z)$  is  $\ell$ -good with torsion free abelianization  $\pi_1^{\ell, \text{ab}}(Z)$  of the  $\ell$ -completion.

**Proof.** By Cor. 2.8 (ii), we may assume that  $Z$  is even smooth and projective. If  $\text{char}(K) = 0$ , then  $\pi_1(Z)$  is  $\ell$ -good as finitely generated fundamental group of a Riemann surface. For the same reason,  $\pi_1^{\ell, \text{ab}}(Z)$  is torsion free. For  $\text{char}(K) > 0$ ,  $\pi_1(Z)$  is  $\ell$ -good by [Sti02] Prop. A.4.1. If  $Z/K$  is proper, there is a smooth lift  $\mathfrak{Z}/W(K)$  of  $Z$  to the Witt-ring  $W(K)$  of  $K$  by [SGA71] Exp. III Cor. 7.4. Let  $Z'$  be a geometric generic fibre  $\bar{\eta}^* \mathfrak{Z}$  of  $\mathfrak{Z}$ . Then the  $\ell$ -completed specialization map  $\pi_1^\ell(Z') \rightarrow \pi_1^\ell(Z)$  is an isomorphism, which settles the second claim. In case  $X$  is affine, it is a  $K(\pi, 1)$ -space of cohomological dimension  $\leq 1$  with  $\ell$ -good fundamental group. In particular,  $\pi_1^\ell(Z)$  has cohomological dimension  $\leq 1$ , i.e., is free (see e.g. [NSW08] Prop. 3.5.17).  $\square$

**$K(\pi, 1)$ -models of  $p$ -adic curves** Let  $X$  be a geometrically connected smooth projective curve over the  $p$ -adic field  $k$  admitting a section  $s$  of  $\pi_1(X/k)$ . In particular,  $X$  is a  $K(\pi, 1)$ -space. We want to modify  $X$  and  $s$  until  $X$  admits a proper flat model  $\eta : X \hookrightarrow \mathfrak{X}$  over  $\mathfrak{o}$  that is a  $K(\pi, 1)$ -space, too. By [SGA71] Exp. X Thm. 2.1 and proper base change, the étale homotopy type of  $\mathfrak{X}$  is weakly equivalent to the homotopy type of its reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}$ . Using Cor. 2.8 (i),  $\mathfrak{X}$  is a  $K(\pi, 1)$  if and only if  $Y$  has no rational components.

Recall that a *neighbourhood*  $(X', s')$  of the section  $s$  is a finite étale covering  $f$  together with a compatible section  $s'$ :

$$\begin{array}{ccc} & \widehat{\text{Et}}(X'/k) & \\ & \uparrow s' & \downarrow f \\ BG_k & \xrightarrow{s} & \widehat{\text{Et}}(X/k). \end{array}$$

**2.11 Lemma.** *Let  $X$  be a geometrically connected smooth projective curve over  $k$  admitting a section  $s$  of  $\pi_1(X/k)$ . Then there is a finite extension  $k'/k$  and a neighbourhood  $(X', s')$  of the restricted section  $s \otimes_k k'$  of  $X \otimes_k k'$  s.t.  $X'$  admits a (stable)  $K(\pi, 1)$ -model over the normalization  $\mathfrak{o}'$  of  $\mathfrak{o}$  in  $k'/k$ .*

**Proof.** By the Stable Reduction Theorem of Deligne and Mumford ([DM69] Cor. 2.7) we may assume that  $X$  has split stable reduction. Using [Moc96] Lem. 2.9 (see also [PS14] Lem. 5.3), there is a finite étale covering  $X'' \rightarrow X$ , s.t. the reduced special fibre of the stable model  $\mathfrak{X}''/\mathfrak{o}''$  of  $X''$  does not admit any rational components. Here,  $\mathfrak{o}''/\mathfrak{o}$  is the normalization inside the field of constants  $k''$  of  $X''$ , i.e.,  $X''$  is geometrically connected over  $k''$ . Then  $s(G_k) \cap \pi_1(X'') \leq s(G_k)$  is open and closed, i.e. of the form  $s(G_{k'})$  for a suitable finite extension  $k'/k$ . The restriction of  $s$  to  $G_{k'}$  factors through  $\pi_1(X'')$ . Composed with the canonical map to  $G_{k''}$ , we get that  $k'$  is an extension of  $k''$ . By construction,  $X' := X'' \otimes_{k''} k'$  is geometrically connected over  $k'$  and  $s$  restricts to a section of  $\pi_1(X'/k')$ , compatible with the restricted section of  $\pi_1(X \otimes_k k'/k')$ . By the Hurwitz formula, the special fibre of  $\mathfrak{X}'' \otimes_{\mathfrak{o}''} \mathfrak{o}'$  does not contain any rational components, i.e.,  $\mathfrak{X}'' \otimes_{\mathfrak{o}''} \mathfrak{o}'$  is a  $K(\pi, 1)$ -model. Finally, as a base change of a stable model, it is still stable.  $\square$

### 3 Specialized sections and homotopy rational points

**Specialized homotopy rational points.** Fix a geometrically connected smooth projective curve  $X$  over  $k$  of genus  $\geq 2$  and a proper flat model  $\eta : X \hookrightarrow \mathfrak{X}$  over  $\mathfrak{o}$  with reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}$ . Say,  $\pi_1(X/k)$  admits a section  $s$  (i.e., a homotopy rational point  $s : BG_k \rightarrow X$ ), unramified with respect to  $\eta$ , i.e., with trivial ramification map

$$\text{ram}_s : I_k \hookrightarrow G_k \xrightarrow{s} \pi_1(X) \xrightarrow{\text{sp}} \pi_1(Y).$$

This is equivalent to  $s$  specializing to a section  $\bar{s}$  of  $\pi_1(Y/\mathbb{F})$ , compatible via the specialization maps.

Combining proper base change with [SGA71] Exp. X Thm. 2.1, we get a specialization map  $\text{sp} : \widehat{\text{Et}}(X/k) \rightarrow \widehat{\text{Et}}(Y/\mathbb{F})$  of étale homotopy types in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$ . Unfortunately, as soon as  $Y$  admits rational components, it is no longer a  $K(\pi, 1)$ -space. So sections of  $\pi_1(Y/\mathbb{F})$  (compatible with section of  $\pi_1(X/k)$ ) do no longer a priori correspond to homotopy rational points  $BG_{\mathbb{F}} \rightarrow \widehat{\text{Et}}(Y/\mathbb{F})$  over  $\mathbb{F}$  (compatible with homotopy rational points of  $X$  over  $k$  via  $\text{sp}$ ). E.g., it is no longer clear if  $\bar{s}$  induces a map on cohomology (compatible with  $s$ ). Thus, we define:

**3.1 Definition.** *Let  $X/k$  be a geometrically connected smooth projective curve over a  $p$ -adic field  $k$  of genus  $\geq 2$  and  $\mathfrak{X}/\mathfrak{o}$  a normal, proper, flat model with reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}/\mathbb{F}$ . We say that a homotopy rational point  $r : BG_k \rightarrow \widehat{\text{Et}}(X/k)$  over  $k$  **specializes** to a homotopy rational point  $\bar{r} : BG_{\mathbb{F}} \rightarrow \widehat{\text{Et}}(Y/\mathbb{F})$  over  $\mathbb{F}$ , if  $r$  and  $\bar{r}$  are compatible via the*

specialization morphism of homotopy types, i.e., if

$$\begin{array}{ccc} BG_k & \xrightarrow{r} & \widehat{\mathrm{Et}}(X/k) \\ \downarrow \text{can.} & & \downarrow \text{sp} \\ BG_{\mathbb{F}} & \xrightarrow{\bar{r}} & \widehat{\mathrm{Et}}(Y/\mathbb{F}) \end{array} \quad (3.1)$$

commutes in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$ . We say that  $r$  **specializes in cohomological settings** to  $\bar{r}$ , if (3.1) induces a commutative square of cohomology cochains  $C^\bullet(-, \Lambda)$  in the derived category  $\mathcal{D}(\underline{\mathbf{Ab}})$  for  $\Lambda$  any continuous finite  $\mathbb{Z}[G_{\mathbb{F}}]$ -module.<sup>2</sup>

**3.2 Remark.** If a homotopy rational point of  $X/k$ , i.e., a section, specializes to a homotopy rational point of  $Y/\mathbb{F}$ , the induced section of  $\pi_1(Y/\mathbb{F})$  is the unique specialization of the original section. In particular, the original section is unramified.

The converse is true if  $Y$  is a  $K(\pi, 1)$ -space. By Lem. 2.11, after a base extension  $k'/k$  an (unramified) section  $s$  has a neighbourhood  $(X', s')$  admitting a  $K(\pi, 1)$ -model. However, it is not known if  $s'$  or even the base extension  $s \otimes_k k'$  is still unramified.

**A candidate for a specialized homotopy rational point.** Suppose all points of the normalization  $\tilde{Y}$  over singular points of  $Y$  contained in rational components are  $\mathbb{F}$ -rational. This is always true after an unramified base extension  $k'/k$ . Note that the section  $s \otimes_k k'$  of  $X \otimes_k k'$  induced by an unramified section  $s$  is still unramified. If  $Y$  admits rational components, it is no longer a  $K(\pi, 1)$ . Thus, a priori the homotopy rational points of  $\widehat{\mathrm{Et}}(Y/\mathbb{F})$  might no longer correspond to sections of  $\pi_1(Y/\mathbb{F})$  modulo conjugation. However, such a canonical correspondence still holds for  $Y/\mathbb{F}$  or more generally for any curve  $Z/K$  as in Thm. 2.4 (iii) and  $K$  of cohomological dimension  $\leq 1$ . Indeed, this is just Cor. 1.6, where a section of  $\widehat{\mathrm{Et}}(Z/K) \rightarrow B\pi_1(Z)$  is provided by Thm. 2.4 (iii):

**3.3 Corollary.** Let  $Z/K$  be a curve as in Thm. 2.4 (iii) for  $K$  of cohomological dimension  $\leq 1$ . Then there is a canonical one-to-one correspondence between sections of  $\pi_1(Z/K)$  modulo conjugation and homotopy rational points  $[BG_K, \widehat{\mathrm{Et}}(Z/K)]_{\hat{\mathcal{S}} \downarrow BG_K}$  of  $Z$ .

**3.4 Remark.** Thus, for a specialized section  $\bar{s}$  there is at least a unique candidate for a specialized homotopy rational point. In abuse of notation, we will denote it by  $\bar{s}$ , as well. Although the compatibility with the specialization maps of homotopy types is unclear for non- $K(\pi, 1)$ -models, we will show that, as a homotopy rational point,  $s$  specializes to  $\bar{s}$  in cohomological settings in Thm. 4.1, below.

**Geometric pro- $\ell$  completion and specialization.** At least, sections always specializes to sections of the geometric pro- $\ell$  completed fundamental group sequence  $\pi^{(\ell)}(Y/\mathbb{F})$ :

**3.5 Lemma.** Any (not necessarily unramified) section  $s$  of  $\pi_1(X/k)$  specializes to a unique section  $\bar{s}_\ell$  of  $\pi_1^{(\ell)}(Y/\mathbb{F})$ , i.e., we have a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{s} & \pi_1(X) \\ \downarrow \text{can.} & & \downarrow \text{sp} \\ G_{\mathbb{F}} & \xrightarrow{\bar{s}_\ell} & \pi_1^{(\ell)}(Y). \end{array}$$

<sup>2</sup>Of course, more generally Def. 3.1 makes sense for  $X$  a proper  $k$ -variety.

**Proof.** This is a direct consequence of [Sti12] Prop. 91: The ramification map  $\text{ram}_s$  induces the geometrically pro- $\ell$  completed ramification map  $\text{ram}_s^{(\ell)} : I_k \rightarrow \pi_1^{(\ell)}(Y)$ . By construction,  $\text{ram}_s^{(\ell)}$  factors over  $\pi_1^\ell(Y \otimes_{\mathbb{F}} \mathbb{F}^s)$ , i.e.,  $\text{ram}_s^{(\ell)}$  is trivial by loc. cit.<sup>3</sup>.  $\square$

**3.6** Suppose all points in  $\tilde{Y}$  lying over singular points of  $Y$  contained in rational components are  $\mathbb{F}$ -rational. Let  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \mathcal{Y}^{(\ell)}$  be the geometric pro- $\ell$  completion  $\widehat{\text{Et}}(Y/\mathbb{F})_{(\ell)}^\wedge$ . By Cor. 2.10,  $\pi_1(Y \otimes_{\mathbb{F}} \mathbb{F}^s)$  is  $\ell$ -good, so  $B\pi_1^{(\ell)}(Y) \simeq (B\pi_1(Y))_{(\ell)}^\wedge$  holds by 1.12. In particular, Thm. 2.4 (iii) gives a section of  $\mathcal{Y}^{(\ell)} \rightarrow B\pi_1^{(\ell)}(Y)$ . The proof of Cor. 3.3 goes through for  $\mathcal{Y}^{(\ell)}$ , i.e., there is a canonical one-to-one correspondence between sections of  $\pi_1^{(\ell)}(Y/\mathbb{F})$  modulo conjugation and homotopy rational points  $[BG_{\mathbb{F}}, \mathcal{Y}^{(\ell)}]_{\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}}$ .

From Lem. 3.5 we get:

**3.7 Corollary.** *For any (not necessarily unramified) section  $s$  of  $\pi_1(X/k)$  there a unique homotopy rational point  $\bar{s}_\ell : BG_{\mathbb{F}} \rightarrow \mathcal{Y}^{(\ell)}$  of  $\mathcal{Y}^{(\ell)}$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$  inducing the specialized section  $\bar{s}_\ell$  of  $\pi_1^{(\ell)}(Y/\mathbb{F})$  in Lem. 3.5.*

Comparing the respective Hochschild-Serre spectral sequences, we get that  $\pi_1(Y) \rightarrow \pi_1^{(\ell)}(Y)$  induces an isomorphism on cohomology with continuous finite  $\mathbb{Z}_\ell[G_{\mathbb{F}}]$ -module coefficients. Luckily, the same is true for  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \mathcal{Y}^{(\ell)}$ :

**3.8 Lemma.** *Let  $\Lambda$  a continuous finite  $\mathbb{Z}_\ell[G_{\mathbb{F}}]$ -module. Then  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \mathcal{Y}^{(\ell)}$  induces an isomorphism*

$$H^\bullet(\mathcal{Y}^{(\ell)}, \Lambda) \xrightarrow{\sim} H^\bullet(Y, \Lambda) .$$

**Proof.**  $\widehat{\text{Et}}(Y/\mathbb{F}) \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}} \rightarrow (\widehat{\text{Et}}(Y/\mathbb{F}) \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}})_\ell^\wedge$  induces an isomorphism on cohomology cochains  $C^\bullet(-, \Lambda)$  in  $\mathcal{D}^+(\text{Mod}_{G_{\mathbb{F}}})$  (see 1.11). Thus, it induces an isomorphism between the respective Hochschild-Serre spectral sequences.  $\square$

**3.9 Remark.** As an alternative to the construction of  $\mathcal{Y}^{(\ell)}$ , we might also work with the following explicit “quasi pro- $\ell$  completion”  $\mathcal{Y}_{\text{alt}}^{(\ell)}$ : By Thm. 2.4 (iii),  $\widehat{\text{Et}}(Y/\mathbb{F})$  is weakly equivalent to  $B\pi_1(Y) \vee_{\coprod_i s_i, \coprod_i y_i} (\coprod_{i \in \mathcal{R}} \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}))$  in  $\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}$  for  $s_i$  suitable sections of  $\pi_1(Y/\mathbb{F})$  and  $y_i$  arbitrary  $\mathbb{F}$ -rational points in the rational components  $\tilde{Y}_i$ . Let  $s_i^{(\ell)}$  be the section of  $\pi_1^{(\ell)}(Y/\mathbb{F})$  induced by  $s_i$ . Then, we get a canonical morphism in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$

$$\widehat{\text{Et}}(Y/\mathbb{F}) \longrightarrow B\pi_1^{(\ell)}(Y) \vee_{\coprod_i s_i^{(\ell)}, \coprod_i y_i} (\coprod_{i \in \mathcal{R}} \tilde{Y}_i) =: \mathcal{Y}_{\text{alt}}^{(\ell)} .$$

Cor. 3.7 and Lem. 3.8 go through for  $\mathcal{Y}_{\text{alt}}^{(\ell)}$ , too.

**Application:  $\text{cl}_s$  and specialized sections.** We want to give a new proof for Prop. B, i.e., [EW09] Cor. 3.4, based on the work done so far.<sup>4</sup> Since the cup product  $\text{cl}_s \cup \alpha$  equals  $s^* \alpha$  for all classes  $\alpha$  in  $H^2(X, \mathbb{Z}_\ell(1))$  (see [Sti12] Sect. 6.1), Tate-Lichtenbaum duality shows that  $\text{cl}_s$  in  $H^2(X, \mathbb{Z}_\ell(1))$  lies in the image of the  $\ell$ -adic Chern class map  $\hat{c}_1 : \text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow H^2(X, \mathbb{Z}_\ell(1))$ , if and only if the composition with  $s^*$  is the trivial map  $\text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow H^2(G_k, \mathbb{Z}_\ell(1))$ . Thus, Prop. B is equivalent to:

<sup>3</sup>Note that [Sti12] Prop. 91 essentially follows from the Weil conjectures.

<sup>4</sup>As Esnault and Wittenberg’s original proof, our proof will make essential use of Deligne’s theory of weights, too (via Lem. 3.5).



**3.10 Proposition.** (cf. [EW09] Cor. 3.4) *Let  $X$  be a geometrically connected smooth projective curve of genus  $\geq 2$  over a  $p$ -adic field  $k$ , admitting a section  $s$  of  $\pi_1(X/k)$ . Then for  $[\mathcal{L}] \in \text{Pic}(X)$  and  $\ell \neq p$ ,  $s^* \hat{c}_1[\mathcal{L}] = 0$  in  $H^2(G_k, \mathbb{Z}_\ell(1))$ .*

**Proof.** Using Lem. 3.5, we get the commutative diagram on cohomology

$$\begin{array}{ccccccc}
 \text{Pic}(X) & \xrightarrow{\hat{c}_1} & H^2(X, \mathbb{Z}_\ell(1)) & \xleftarrow{\cong} & H^2(\pi_1(X), \mathbb{Z}_\ell(1)) & \xrightarrow{s^*} & H^2(G_k, \mathbb{Z}_\ell(1)) \\
 \uparrow \eta^* & & \uparrow \eta^* & & \uparrow & \searrow & \uparrow \\
 \text{Pic}(\mathfrak{X}) & \xrightarrow{\hat{c}_1} & H^2(\mathfrak{X}, \mathbb{Z}_\ell(1)) & \xleftarrow{\quad} & H^2(\pi_1(\mathfrak{X}), \mathbb{Z}_\ell(1)) & & \\
 & & \downarrow \sigma^* & & \downarrow \cong & \nearrow & \\
 & & H^2(Y, \mathbb{Z}_\ell(1)) & \xleftarrow{\quad} & H^2(\pi_1(Y), \mathbb{Z}_\ell(1)) & & \\
 & & & & \uparrow \cong & & \\
 & & & & H^2(\pi_1^{(\ell)}(Y), \mathbb{Z}_\ell(1)) & \xrightarrow{\bar{s}_\ell^*} & H^2(G_{\mathbb{F}}, \mathbb{Z}_\ell(1)).
 \end{array} \tag{3.2}$$

Since  $\mathbb{F}$  has cohomological dimension 1,  $H^2(G_{\mathbb{F}}, \mathbb{Z}_\ell(1))$  is trivial. Thus, Prop. 3.10 would follow if all horizontal leftward maps were isomorphisms (i.e., if  $Y$  is a  $K(\pi, 1)$ -space) and if  $[\mathcal{L}]$  lied in the image of  $\eta^*$ . To guarantee this, note that we may replace  $[\mathcal{L}]$  by a multiple  $[\mathcal{L}^{\otimes n}]$  and enlarge the base field  $k$  by any finite extension  $k'/k$ : Indeed,  $H^2(G_k, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell$  is torsion free, so a non-zero class  $s^* \hat{c}_1[\mathcal{L}]$  would stay non-trivial after multiplication by  $n$  resp. base extension to  $k'$ . It is clear that we might replace the pair  $(X, s)$  by a neighbourhood  $(X', s')$ . Thus, by Lem. 2.11, we might indeed assume that  $Y$  is a  $K(\pi, 1)$ -space and  $\mathfrak{X}$  the stable model of  $X$ . As such,  $\mathfrak{X}$  has at most rational singularities (this follows from [Lip69] Thm. 27.1, use that we get  $\mathfrak{X}$  by contracting rational curves in the minimal regular model with self intersection  $-2$ ). These are  $\mathbb{Q}$ -factorial by loc. cit. §17, i.e., for suitable  $n \gg 0$ , the  $n$ -th multiple of the closure in  $\mathfrak{X}$  of the Weil-divisor corresponding to  $[\mathcal{L}]$  is a Cartier-divisor and thus a pre-image for  $[\mathcal{L}^{\otimes n}]$  under  $\eta^*$ .  $\square$

**3.11 Remark.** Let  $\mathfrak{X}$  be a regular (not necessarily  $K(\pi, 1)$ -) model, s.t.  $Y$  satisfies the assumptions of Thm. 2.4 (iii). It will turn out later that the homotopy rational point  $s$  “specializes in cohomological settings” to the homotopy rational point  $\bar{s}_\ell$  of  $\mathcal{Y}^{(\ell)}$  given by Cor. 3.7 (see Cor. 4.9, below). Thus, Prop. 3.10 would follow directly from Lem. 3.8 and a diagram chase in (3.2). Unfortunately, we don’t know how to prove Cor. 4.9 without using Prop. 3.10 itself.

**3.12 Remark.** Let  $f : X \rightarrow \mathbb{P}_k^N$  be a non-constant  $k$ -morphism,  $\mathcal{L} = f^* \mathcal{O}(1)$  and  $r = f \circ s$  resp.  $r_\infty$  the induced homotopy rational point of  $\mathbb{P}_k^N/k$  resp.  $\mathbb{P}_k^\infty/k$  - here we embed  $\mathbb{P}_k^N$  into  $\mathbb{P}_k^\infty$  via the first  $N + 1$  coordinate functions. The homotopy type of  $\mathbb{P}_{\mathbb{F}}^\infty$  represents  $H^2(-, \hat{\mathbb{Z}}^{(p')}(1))$  in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$  and the composition of  $r_\infty$  with the specialization morphism of homotopy types  $\widehat{\text{Et}}(\mathbb{P}_k^\infty/k) \rightarrow \widehat{\text{Et}}(\mathbb{P}_{\mathbb{F}}^\infty/\mathbb{F})$  corresponds to the class  $s^* \hat{c}_1[\mathcal{L}]$  in  $H^2(G_k, \hat{\mathbb{Z}}^{(p')}(1))$ . Since  $H^2(G_{\mathbb{F}}, \hat{\mathbb{Z}}^{(p')}(1))$  is trivial, Prop. 3.10 implies that the homotopy rational point  $r_\infty$  specializes to the unique homotopy rational point of  $\mathbb{P}_{\mathbb{F}}^\infty$ . The condition that  $s^* \hat{c}_1[\mathcal{L}]$  (or more generally  $r^* \hat{c}_1[\mathcal{M}]$  for  $r$  a homotopy rational point of a Brauer-Severi variety over  $k$  and  $\mathcal{M}$  a non-trivial line bundle) is trivial even in  $H^2(G_k, \hat{\mathbb{Z}}(1))$  implies that  $r$  is homotopic to a rational point of  $\mathbb{P}_k^N$  (see [Sch15] Thm. 4.3).

**3.13 Remark.** Suppose  $s$  trivializes  $\hat{c}_1$  even in  $H^2(G_k, \hat{\mathbb{Z}}(1))$ . By Tate-Lichtenbaum duality, this is equivalent to the algebraicity of the cycle class  $\text{cl}_s$ . In particular,  $X$  admits an algebraic cycle of degree 1 in this case. In this sense, such a section  $s$  satisfies a *linearized form of the  $p$ -adic*

*weak section conjecture* (which clearly is predicted by the  $p$ -adic weak section conjecture). By Roquette-Lichtenbaum, this is equivalent to the triviality of the relative Brauer group  $\mathrm{Br}(X/k)$ . For a general section  $s$ , Prop. B implies that  $\mathrm{Br}(X/k)$  is a  $p$ -group, which was first proved by Stix (see [Sti10] Thm. 15) using different methods and including even the genus 1 case.

## 4 Specialized sections and cohomology

**Specialized homotopy rational points in cohomological settings.** Fix a geometrically connected smooth projective curve  $X$  over  $k$  of genus  $\geq 2$  admitting a section  $s$  unramified with respect to the regular proper flat model  $\eta : X \hookrightarrow \mathfrak{X}$  over  $\mathfrak{o}$  with reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\mathrm{red}}$ . Again, we assume that all points of  $\tilde{Y}$  lying over singular points of  $Y$  contained in rational components are  $\mathbb{F}$ -rational. Then  $s$  specializes in cohomological settings (the proof will show more generally: in “sufficiently additive settings”) to the homotopy rational point  $\bar{s}$  of  $Y/\mathbb{F}$  given by Cor. 3.3:

**4.1 Theorem.** *Let  $X/k$  be a geometrically connected smooth projective curve over a  $p$ -adic field  $k$  of genus  $\geq 2$  and  $\mathfrak{X}/\mathfrak{o}$  a regular, proper, flat model s.t. the reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\mathrm{red}}/\mathbb{F}$  satisfies the assumptions of Thm. 2.4 (iii). For an unramified section  $s$ , its specialized section  $\bar{s}$  and a constructible  $G_{\mathbb{F}}$ -module  $\Lambda$ , (3.1) induces a commutative diagram of cohomology cochains in the derived category  $\mathcal{D}^+(\underline{\mathrm{Ab}})$  resp.  $\mathcal{D}^+(\underline{\mathrm{Mod}}_{G_{\mathbb{F}}})$ :*

$$\begin{array}{ccc} C^\bullet(X, \Lambda) & \xrightarrow{s^*} & C^\bullet(G_k, \Lambda) \\ \uparrow \mathrm{sp}^* & & \uparrow \\ C^\bullet(Y, \Lambda) & \xrightarrow{\bar{s}^*} & C^\bullet(G_{\mathbb{F}}, \Lambda) \end{array} \quad \text{resp.} \quad \begin{array}{ccc} C^\bullet(X \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}}, \Lambda) & \xrightarrow{s^*} & C^\bullet(BG_k \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}}, \Lambda) \\ \uparrow \mathrm{sp}^* & & \uparrow \\ C^\bullet(Y \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}}, \Lambda) & \xrightarrow{\bar{s}^*} & C^\bullet(EG_{\mathbb{F}}, \Lambda). \end{array}$$

To prove Thm. 4.1 we try to work out what the obstruction for the commutativity could be:

**4.2** By Thm. 2.4 (iii),  $\widehat{\mathrm{Et}}(Y/\mathbb{F}) \simeq (B\pi_1(Y)) \vee \coprod_i s_i, \coprod_i y_i (\coprod_{i \in \mathcal{R}} \widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F}))$  holds in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow G_{\mathbb{F}})$  for any  $\mathbb{F}$ -points  $y_i$  of  $\tilde{Y}_i \cong \mathbb{P}_{\mathbb{F}}^1$ . Contracting the  $K(\pi, 1)$ -part  $B\pi_1(Y)$  in  $\hat{\mathcal{S}} \downarrow G_{\mathbb{F}}$  gives a homotopy co-cartesian square

$$\begin{array}{ccc} B\pi_1(Y) & \xrightarrow{\quad} & BG_{\mathbb{F}} \\ \downarrow \iota & & \downarrow * \\ (B\pi_1(Y)) \vee \coprod_i s_i, \coprod_i y_i (\coprod_{i \in \mathcal{R}} \widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F})) & \xrightarrow{p} & \bigvee_{i \in \mathcal{R}}^{i \in \mathcal{R}} (\widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F}), y_i) \end{array}$$

in  $\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}$ , i.e., an exact triangle in  $\mathcal{D}^+(\underline{\mathrm{Ab}})$

$$\begin{array}{ccc} C^\bullet(\bigvee_{i \in \mathcal{R}}^{i \in \mathcal{R}} (\widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F}), y_i), \Lambda) & \xrightarrow{p^* \oplus *} & C^\bullet(Y, \Lambda) \oplus C^\bullet(G_{\mathbb{F}}, \Lambda) \\ & \nwarrow +1 & \downarrow \iota^* - \mathrm{can} \\ & & C^\bullet(\pi_1(Y), \Lambda). \end{array} \quad (4.1)$$

The section  $s$  together with  $\mathrm{sp}$  and the “projection”  $p$  induces a morphism

$$BG_k \xrightarrow{s} \widehat{\mathrm{Et}}(X/k) \xrightarrow{:=\alpha} \mathrm{cosk}_3 \bigvee_{i \in \mathcal{R}}^{i \in \mathcal{R}} (\widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F}), y_i) \quad (4.2)$$

in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$ . Assume that it factors as the canonical morphism to  $BG_{\mathbb{F}}$  followed by the distinguished point  $*$  :  $BG_{\mathbb{F}} \rightarrow \mathrm{cosk}_3 \bigvee_{i \in \mathcal{R}}^{i \in \mathcal{R}} (\widehat{\mathrm{Et}}(\tilde{Y}_i/\mathbb{F}), y_i)$ . Truncation in degrees  $> 3$  gives the

commutative diagram

$$\begin{array}{ccc}
C^\bullet(\bigvee_{BG_{\mathbb{F}}}^{i \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), y_i), \Lambda) & \xrightarrow{\quad} & C^\bullet(G_k, \Lambda) \\
& \searrow * & \nearrow \text{can} \\
& C^\bullet(G_{\mathbb{F}}, \Lambda) &
\end{array}$$

on cochains: Indeed,  $k$  and  $\mathbb{F}$  have cohomological dimension  $\leq 2$  and  $C^\bullet(\bigvee_{BG_{\mathbb{F}}}^{i \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), y_i), \Lambda)$  is the truncation  $\tau^{\leq 3} C^\bullet(\text{cosk}_3 \bigvee_{BG_{\mathbb{F}}}^{i \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), y_i), \Lambda)$ , since the rational components geometrically have cohomological dimension 2. It follows that

$$C^\bullet(Y, \Lambda) \oplus C^\bullet(G_{\mathbb{F}}, \Lambda) \xrightarrow{(\text{sp} \circ s)^* - \text{can}} C^\bullet(G_k, \Lambda)$$

extends to a morphism of the exact triangle (4.1) to the trivial triangle

$$0 \longrightarrow C^\bullet(G_k, \Lambda) \xrightarrow{\text{id}} C^\bullet(G_k, \Lambda) \xrightarrow{+1} 0[1].$$

In particular,  $C^\bullet(Y, \Lambda) \rightarrow C^\bullet(G_k, \Lambda)$  factors through  $\iota^*$ . Since  $\iota^*$  is a retraction of the canonical morphism  $C^\bullet(\pi_1(Y), \Lambda) \rightarrow C^\bullet(Y, \Lambda)$ , the resulting morphism  $C^\bullet(\pi_1(Y), \Lambda) \rightarrow C^\bullet(G_k, \Lambda)$  is just the morphism given by the original section  $s$  and  $\text{sp}$  on fundamental groups. Combining this with the compatibility of  $s$  with the specialized section  $\bar{s}$ , we get the commutative diagram

$$\begin{array}{ccccc}
C^\bullet(X, \Lambda) & \xrightarrow{s^*} & C^\bullet(BG_k, \Lambda) & & \\
\uparrow \text{sp}^* & & \nearrow (\text{sp} \circ s)^* & & \uparrow \text{can} \\
C^\bullet(Y, \Lambda) & \xrightarrow{\iota^*} C^\bullet(\pi_1(Y), \Lambda) & \xrightarrow{\bar{s}^*} C^\bullet(BG_{\mathbb{F}}, \Lambda), & & 
\end{array}$$

where the “outer” commutative square is the square induced by (3.1). Of course, the same arguments work in the  $G_{\mathbb{F}}$ -equivariant setting, as well.

Let us sum up 4.2: To prove Thm. 4.1, it is enough to show that the composed morphism (4.2) factors in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$  as the canonical morphism to  $BG_{\mathbb{F}}$  followed by the “distinguished point”  $*$ :  $BG_{\mathbb{F}} \rightarrow \text{cosk}_3 \bigvee_{BG_{\mathbb{F}}}^{i \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), y_i)$ . This problem can be translated into the vanishing of a certain cohomology class:

**4.3** Under the Quillen-equivalence of  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$  and  $\mathcal{H}(\hat{\mathcal{S}}_{G_{\mathbb{F}}})$ ,  $\text{cosk}_3 \bigvee_{BG_{\mathbb{F}}}^{i \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), y_i)$  corresponds to a  $K(\bigoplus_{i \in \mathcal{R}} \hat{\mathbb{Z}}^{(p')}(1), 2)$ :  $\tilde{Y}_i$  is isomorphic to  $\mathbb{P}_{\mathbb{F}}^1$ , hence geometrically simply connected and  $\mathbb{P}_{\mathbb{F}}^1 \otimes_{\mathbb{F}} \mathbb{F}^s$  has cohomological- $p$ -dimension  $\leq 1$  (in fact  $= 0$ ) as projective curve over a separably closed field of characteristic  $p$ . Denote by  $\alpha_i$  the  $i^{\text{th}}$ -summand in  $H^2(X, \hat{\mathbb{Z}}(1)^{(p')})$  of the class corresponding to  $\alpha$  in the composition (4.2). Since  $G_{\mathbb{F}}$  has cohomological dimension 1, our factorization problem translates to the question whether  $s^* \alpha_i$  is trivial in  $H^2(G_k, \hat{\mathbb{Z}}(1)^{(p')})$ . As prime-to- $p$  Tate module of the Brauer group,  $H^2(G_k, \hat{\mathbb{Z}}(1)^{(p')})$  is canonically isomorphic to  $\hat{\mathbb{Z}}^{(p')}$  and it remains to show that  $s^* \alpha_i$  vanishes in  $H^2(G_k, \mathbb{Q}_{\ell}(1))$  for all  $i \in \mathcal{R}$  and  $\ell \neq p$ .

We want to have a better understanding where the classes  $\alpha_i$  come from:

**4.4** As a morphism in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$ ,  $\alpha_i$  in  $H^2(X, \hat{\mathbb{Z}}(1)^{(p')})$  is given as the 3-coskeleton of the composition

$$\widehat{\text{Et}}(X/k) \xrightarrow{\text{sp}} \widehat{\text{Et}}(Y/\mathbb{F}) \xrightarrow{p} \bigvee_{BG_{\mathbb{F}}}^{j \in \mathcal{R}} (\widehat{\text{Et}}(\tilde{Y}_j/\mathbb{F}), y_j) \longrightarrow \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}), \quad (4.3)$$

where the last morphism is given by contracting all  $\widehat{\text{Et}}(\tilde{Y}_j/\mathbb{F})$  for  $j \neq i$  to  $BG_{\mathbb{F}}$ . Let  $\beta_i$  be the class in  $H^2(Y, \hat{\mathbb{Z}}^{(p')}(1))$  (resp. in  $H^2(Y, \mathbb{Q}_{\ell}(1))$ ) given by the morphism  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$  in between composed with  $\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \rightarrow \text{cosk}_3 \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$ , i.e.,  $\alpha_i = \text{sp}^* \beta_i$ . Since  $\text{cosk}_3 \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$  represents  $H^2(-, \hat{\mathbb{Z}}^{(p')}(1))$  and  $\text{cosk}_3$  does not change  $H^2$ ,  $\beta_i$  in  $H^2(Y, \mathbb{Q}_{\ell}(1))$  is given as the image of 1 in  $H^2(\tilde{Y}_i, \mathbb{Q}_{\ell}(1)) = \mathbb{Q}_{\ell}$  under the morphism  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$  in the composition (4.3).

**4.5 Remark.** Note that the vanishing of  $s^* \alpha_i$  in fact is necessary: In order for Thm. 4.1 to hold, it is necessary for the class  $s^* \alpha_i$  to vanish in  $H^2(G_k, \mathbb{Q}_{\ell}(1))$ : By the usual arguments,  $H^2(G_k, \mathbb{Z}_{\ell}(1))$  and  $H^2(X, \mathbb{Z}_{\ell}(1))$  are given as the limits over the corresponding cohomology groups with coefficients  $\mu_{\ell^n}$ . Since  $\alpha_i = \text{sp}^* \beta_i$  hold by 4.4 and  $\mathbb{F}$  has cohomological dimension 1, the triviality of  $s^* \alpha_i$  would follow from Thm. 4.1.

We want to give a slightly different characterization of the class  $\beta_i$  than the one in 4.4:

**4.6** As in the proof of Thm. 2.4 (iii), let  $Y' \hookrightarrow Y$  be the union of components  $Y_j$  for  $j \notin \mathcal{R}$ ,  $Y'' \rightarrow Y'$  the normalization at all singular points of  $Y$  contained in rational components and  $\mathcal{Y}_{\mathcal{R}}^*$  the homotopy type we get from joining  $\widehat{\text{Et}}(Y''/\mathbb{F})$  to the homotopy types of the remaining rational components  $\tilde{Y}_j$  via the stars of the corresponding singularities. Recall that the projection  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$  is given by  $\widehat{\text{Et}}(Y/\mathbb{F}) \simeq \mathcal{Y}_{\mathcal{R}}^*$ , then moving all points in  $\tilde{Y}_j$  over singular points on  $Y$  to one  $\mathbb{F}$ -rational point  $y_j$  for each  $j \in \mathcal{R}$ , then contracting the  $K(\pi, 1)$ -part  $B\pi_1(Y)$  and finally contracting the other rational components  $\widehat{\text{Et}}(\tilde{Y}_j/\mathbb{F})$  for  $j \neq i$ . We could as well contract the rational components  $\tilde{Y}_j$  for  $j \neq i$  and  $Y''$  in  $\mathcal{Y}_{\mathcal{R}}^*$  first, then moving all points  $z_1, \dots, z_t$  in  $\tilde{Y}_i$  over singular points on  $Y$  to  $y_i$  to get  $\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \vee_{y_i} (BG_{\mathbb{F}} \otimes \bar{\Gamma}(i))$  for a certain contraction  $\Gamma(Y) \rightarrow \bar{\Gamma}(i)$  of the dual graph and then contract the graph  $\bar{\Gamma}(i)$  to get the same morphism  $\mathcal{Y}_{\mathcal{R}}^* \rightarrow \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F})$ . We get the homotopy pushout

$$\mathcal{Y}^{(i)} := \widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \vee_{z_1, \dots, z_t} (BG_{\mathbb{F}} \otimes \Gamma(i))$$

after the first step, where  $\Gamma(i)$  is constructed in the same way as  $\bar{\Gamma}(i)$  but without gluing together the rays of stars corresponding to singularities of  $Y$  in  $Y_i$  and ending in the node corresponding to  $Y_i$  and where we glue the remaining “open” ends of  $BG_{\mathbb{F}} \otimes \Gamma(i)$  to the corresponding points  $z_1, \dots, z_t$  in  $\tilde{Y}_i$ . By Lem. 1.10,  $\mathcal{Y}^{(i)}$  and  $\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \vee_{y_i} (BG_{\mathbb{F}} \otimes \bar{\Gamma}(i))$  are weakly equivalent in  $\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}$ . Further, contracting the graph  $\bar{\Gamma}(i)$  induces an isomorphism  $H^2(\tilde{Y}_i, \mathbb{Q}_{\ell}(1)) \rightarrow H^2(\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \vee_{y_i} (BG_{\mathbb{F}} \otimes \bar{\Gamma}(i)), \mathbb{Q}_{\ell}(1))$ : The target is isomorphic to  $H^2(\tilde{Y}_i, \mathbb{Q}_{\ell}(1)) \oplus H^2(BG_{\mathbb{F}} \otimes \bar{\Gamma}(i), \mathbb{Q}_{\ell}(1))$ , since the co-cartesian square of  $\widehat{\text{Et}}(\tilde{Y}_i/\mathbb{F}) \vee_{y_i} (BG_{\mathbb{F}} \otimes \bar{\Gamma}(i))$  is even homotopy co-cartesian and  $\mathbb{F}$  has cohomological dimension 1. For  $m$  prime to  $p$  we find  $H^2(BG_{\mathbb{F}} \otimes \bar{\Gamma}(i), \mu_m) = (\mathbb{F}^{\times}/(\mathbb{F}^{\times})^m)^{\oplus r}$  for  $r$  the number of loops in  $\Gamma(Y)$  using the Hochschild-Serre spectral sequence. Hence,  $H^2(BG_{\mathbb{F}} \otimes \bar{\Gamma}(i), \mathbb{Z}_{\ell}(1))$  is torsion. Summing up, we get the class  $\beta_i$  in  $H^2(Y, \mathbb{Q}_{\ell}(1))$  as the image of 1 under the map

$$\mathbb{Q}_{\ell} = H^2(\mathcal{Y}^{(i)}, \mathbb{Q}_{\ell}(1)) \longrightarrow H^2(Y, \mathbb{Q}_{\ell}(1)). \quad (4.4)$$

Having made all these preparations, we get Thm. 4.1 as a consequence of Prop. 3.10:

**Proof of Thm. 4.1.** For each rational component  $\tilde{Y}_i$  of  $\tilde{Y}$ , choose a closed point  $x_i$  of  $X$  whose specialization lies in the component  $Y_i$  outside the singular locus of  $Y$  and  $\neq y_i$ . We claim that  $\beta$  generates the same  $\mathbb{Q}_{\ell}$ -subspace in  $H^2(\mathfrak{X}, \mathbb{Q}_{\ell}(1))$  than the Chern class  $\hat{c}_1[\mathcal{O}_{\mathfrak{X}}(D_i)]$  for  $D_i = D$  the divisor we get as the closure of  $x_i$  in  $\mathfrak{X}$ :

By Lem. 4.7, below,  $H_D^2(\mathfrak{X}, \mathbb{Q}_{\ell}(1))$  can be computed as continuous cohomology of the  $BG_{\mathbb{F}}$ -space  $\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X}-D/\mathfrak{o})} BG_{\mathbb{F}}$  (note that  $\widehat{\text{Et}}(\mathfrak{o}, \bar{\sigma}) = BG_{\mathbb{F}}$  since each hypercovering in the étale

cite of  $\mathfrak{o}$  can be refined to a hypercovering in the finite étale site). Let  $Y^\bullet$  be the punctured curve  $Y - \text{sp}(x_i)$  and  $\mathcal{Y}_{\mathcal{R}}^{\bullet,*} \simeq \widehat{\text{Et}}(Y^\bullet/\mathbb{F})$  the analogue construction to  $\mathcal{Y}_{\mathcal{R}}^*$  with  $\tilde{Y}_i$  replaced by the punctured component  $\tilde{Y}_i - \text{sp}(x_i)$ .<sup>5</sup> Then  $\widehat{\text{Et}}(Y/\mathbb{F}) \vee_{\widehat{\text{Et}}(Y^\bullet/\mathbb{F})} BG_{\mathbb{F}}$  is isomorphic to  $\mathcal{Y}_{\mathcal{R}}^* \vee_{\mathcal{Y}_{\mathcal{R}}^{\bullet,*}} BG_{\mathbb{F}}$  by the proof of Thm. 2.4 (iii). Similar to  $\mathcal{Y}^{(i)}$ , we get

$$\mathcal{Y}^{(i),\bullet} := \widehat{\text{Et}}(\tilde{Y}_i - \text{sp}(x_i)/\mathbb{F}) \vee_{z_1, \dots, z_t} (BG_{\mathbb{F}} \otimes \Gamma.(i))$$

after contracting all rational components  $\tilde{Y}_j$  for  $j \neq i$  and  $Y''$  in  $\mathcal{Y}_{\mathcal{R}}^{\bullet,*}$ . Summing up, we get a commutative square

$$\begin{array}{ccccc} \widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o}) & \hookrightarrow & \widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) & \longrightarrow & \widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o})} BG_{\mathbb{F}} \\ \uparrow & & \uparrow \sim & & \uparrow \\ \mathcal{Y}_{\mathcal{R}}^{\bullet,*} & \hookrightarrow & \mathcal{Y}_{\mathcal{R}}^* & \longrightarrow & \mathcal{Y}_{\mathcal{R}}^* \vee_{\mathcal{Y}_{\mathcal{R}}^{\bullet,*}} BG_{\mathbb{F}} \\ \downarrow & & \downarrow & & \downarrow \sim \\ \mathcal{Y}^{(i),\bullet} & \hookrightarrow & \mathcal{Y}^{(i)} & \longrightarrow & \mathcal{Y}^{(i)} \vee_{\mathcal{Y}^{(i),\bullet}} BG_{\mathbb{F}} \end{array}$$

in  $\mathcal{H}(\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}})$ . Observe that  $Y''$  and all the rational components  $\tilde{Y}_j$  for  $j \neq i$  we contracted in  $\mathcal{Y}^{(i)}$  resp.  $\mathcal{Y}^{(i),\bullet}$  factor through  $\mathcal{Y}_{\mathcal{R}}^{\bullet,*}$ , so the lower right vertical morphism is a weak equivalence. It follows that the canonical morphism  $\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \rightarrow \widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o})} BG_{\mathbb{F}}$  factors through  $\mathcal{Y}_{\mathcal{R}}^* \rightarrow \mathcal{Y}^{(i)}$ , i.e., the (non-trivial) cycle class map  $H_D^2(\mathfrak{X}, \mathbb{Q}_\ell(1)) \rightarrow H^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$  factors through (4.4), whose image is generated by  $\beta_i$ .

Now  $\eta^* \hat{c}_1[\mathcal{O}_{\mathfrak{X}}(D_i)] = \hat{c}_1[\mathcal{O}_X(x_i)]$  lies in the kernel of  $s^*$  by Prop. 3.10. Thus, the same is true for  $\alpha_i$  and Thm. 4.1 follows from 4.3 and 4.2.  $\square$

**4.7 Lemma.** *Let  $\mathfrak{X}$  be as in Thm. 4.1,  $\ell \neq p$  a prime and  $D$  a regular prime-divisor of  $\mathfrak{X}$ . Then the continuous cohomology groups  $H^2(\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o})} BG_{\mathbb{F}}, \mathbb{Q}_\ell(1))$  and  $H_D^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$  are canonically isomorphic.*

**Proof.** For  $m$  prime to  $p$  take the (non-canonically given) morphism of exact triangles

$$\begin{array}{ccccccc} C^\bullet(\mathfrak{X}, \mathfrak{X} - D, \mu_m) & \longrightarrow & C^\bullet(\mathfrak{X}, \mu_m) & \longrightarrow & C^\bullet(\mathfrak{X} - D, \mu_m) & \xrightarrow{+1} \\ \downarrow & & \downarrow \text{can} & & \parallel & \\ C^\bullet(\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o})} BG_{\mathbb{F}}, \mu_m) & \longrightarrow & C^\bullet(\mathfrak{X}, \mu_m) \oplus C^\bullet(G_{\mathbb{F}}, \mu_m) & \longrightarrow & C^\bullet(\mathfrak{X} - D, \mu_m) & \xrightarrow{+1} \end{array}$$

where the upper triangle computes  $H_D^\bullet(\mathfrak{X}, \mu_m)$  by [Fri82] Cor. 14.5 and Prop. 14.6 and the lower triangle is given by the homotopy co-cartesian square

$$\begin{array}{ccc} \widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o}) & \longrightarrow & BG_{\mathbb{F}} \\ \downarrow & & \downarrow \\ \widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) & \longrightarrow & \widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X} - D/\mathfrak{o})} BG_{\mathbb{F}} \end{array}$$

<sup>5</sup>Note that  $\mathcal{Y}_{\mathcal{R}}^{\bullet,*} \neq \mathcal{Z}_{\mathcal{R}(Z)}^*$  for  $Z = Y^\bullet$  in the proof of Thm. 2.4 (iii):  $Y_i - \text{sp}(x_i)$  is not projective so it would be a component of  $Z'$ .

in  $\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}$ . Note that the induced maps  $H_D^q(\mathfrak{X}, \mu_m) \rightarrow H^q(\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X}-D/\mathfrak{o})} BG_{\mathbb{F}}, \mu_m)$  are uniquely determined up to canonical isomorphisms. The relevant cohomology groups (i.e.  $H^1$  for  $\mathfrak{X}$ ,  $\mathfrak{X} - D$ ,  $G_{\mathbb{F}}$  and  $H^2$  for  $\mathfrak{X}$ ,  $G_{\mathbb{F}}$ ) are all finite by proper base change and [SGA77] Th. finitude together with the finiteness for  $G_{\mathbb{F}}$ -cohomology of finite prime-to- $p$  modules, resp. for  $H^1(\mathfrak{X} - D, \mu_m)$  by [SGA77] Cycle Prop. 2.1.4. It follows that the resulting short exact sequences computing  $H_D^2(\mathfrak{X}, \mu_m)$  resp.  $H^2(\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X}-D/\mathfrak{o})} BG_{\mathbb{F}}, \mu_m)$  stay exact after taking the limit over all  $m = \ell^n$  and compute the respective continuous cohomology groups (by [Jan88] (3.1)). Everything stays exact after  $(-) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$  and the resulting canonical map  $H_D^2(\mathfrak{X}, \mathbb{Q}_{\ell}(1)) \rightarrow H^2(\widehat{\text{Et}}(\mathfrak{X}/\mathfrak{o}) \vee_{\widehat{\text{Et}}(\mathfrak{X}-D/\mathfrak{o})} BG_{\mathbb{F}}, \mathbb{Q}_{\ell}(1))$  is an isomorphism:  $H^2(G_{\mathbb{F}}, \mathbb{Z}_{\ell}(1))$  is trivial and  $H^1(G_{\mathbb{F}}, \mathbb{Z}_{\ell}(1))$  is a quotient of the multiplicative group  $\mathbb{F}^{\times}$  and hence torsion, since  $H^1(G_{\mathbb{F}}, \mu_m) = \mathbb{F}^{\times}/(\mathbb{F}^{\times})^m$ .  $\square$

**4.8 Remark.** Looking back onto the proof resp. the reductions leading to the proof of Thm. 4.1, we see that essentially we just proved the factorization

$$\begin{array}{ccc} C^{\bullet}(X, \Lambda) & \xrightarrow{s^*} & C^{\bullet}(BG_k, \Lambda) \\ \uparrow \text{sp}^* & & \uparrow (\text{sp} \circ s)^* \\ C^{\bullet}(Y, \Lambda) & \xrightarrow{\iota^*} & C^{\bullet}(\pi_1(Y), \Lambda) \end{array}$$

for any (possibly ramified) section  $s$  of  $\pi_1(X/k)$ , where  $\iota$  was the homotopy splitting of the canonical morphism  $\widehat{\text{Et}}(Y/\mathbb{F}) \rightarrow B\pi_1(Y)$  given by Thm. 2.4 (iii).

Combining Cor. 3.7 and Lem. 3.8 with the observation in Rem. 4.8 implies:

**4.9 Corollary.** *Let  $X/k$  be a geometrically connected smooth projective curve over a  $p$ -adic field  $k$  of genus  $\geq 2$ ,  $\mathfrak{X}/\mathfrak{o}$  a regular, proper, flat model s.t. the reduced special fibre  $Y/\mathbb{F}$  satisfies the assumptions of Thm. 2.4 (iii). Then for any section  $s$  of  $\pi_1(X/k)$  there is a unique homotopy rational point  $\bar{s}_{\ell}$  in  $[BG_{\mathbb{F}}, \mathcal{Y}^{(\ell)}]_{\hat{\mathcal{S}} \downarrow BG_{\mathbb{F}}}$  inducing commutative diagrams*

$$\begin{array}{ccccc} G_k \xrightarrow{s} \pi_1(X) & & C^{\bullet}(X, \Lambda) \xrightarrow{s^*} C^{\bullet}(G_k, \Lambda) & & C^{\bullet}(X \otimes_k k^{\text{nr}}, \Lambda) \xrightarrow{s^*} C^{\bullet}(I_k, \Lambda) \\ \downarrow & \text{and} & \uparrow \text{sp}^* & \text{resp.} & \uparrow \text{sp}^* \\ G_{\mathbb{F}} \xrightarrow{\bar{s}_{\ell}} \pi_1^{(\ell)}(Y) & & C^{\bullet}(Y, \Lambda) \xrightarrow{\bar{s}_{\ell}^*} C^{\bullet}(G_{\mathbb{F}}, \Lambda) & & C^{\bullet}(Y^s, \Lambda) \xrightarrow{\bar{s}_{\ell}^*} \Lambda \end{array}$$

in the category of profinite groups resp.  $\mathcal{D}^+(\underline{\text{Ab}})$  resp.  $\mathcal{D}^+(\underline{\text{Mod}}_{G_{\mathbb{F}}})$  for  $\Lambda$  any continuous finite  $\mathbb{Z}_{\ell}[G_{\mathbb{F}}]$ -module.  $\square$

**Application: A canonical lift of  $\text{cl}_s$ .** In [EW09] Rem. 3.7 (iii) Esnault and Wittenberg raised the question whether the  $\ell$ -adic cycle class  $\text{cl}_s$  of a section  $s$  of  $\pi_1(X/k)$  admits a canonical lift to  $H^2(\mathfrak{X}, \mathbb{Z}_{\ell}(1))$ . This is predicted by the  $p$ -adic section conjecture: If  $s$  comes from a rational point  $x$ ,  $\text{cl}_s$  is given by the Chern class of the corresponding divisor  $D = x$  and the Chern class of its closure is a canonical lift. As an application of Cor. 4.9, we will sketch the construction of a canonical lift of  $\text{cl}_s$  to  $H^2(\mathfrak{X}, \mathbb{Z}_{\ell}(1))$  for  $\mathfrak{X}/\mathfrak{o}$  a regular, proper, flat model s.t. the reduced special fibre satisfies the assumptions of Thm. 2.4 (iii). First, let us shortly recall the definition of the cycle class  $\text{cl}_s$ :

**4.10 Remark.** Let  $X_s \rightarrow X$  be the universal neighbourhood of the section  $s$  and  $\Delta_X$  the diagonal in  $X \times_k X$ . Then  $H^2(X, \mu_{\ell^n})$  is isomorphic to  $H^2(X \times_k X_s, \mu_{\ell^n})$  and  $\text{cl}_s$  corresponds to the pullback of  $\hat{c}_1[\Delta_X]$  along  $X \times_k X_s \rightarrow X \times_k X$ . Since  $X$  is a  $K(\pi, 1)$ ,  $X_s \rightarrow X$  is



weakly equivalent to  $s$  as a homotopy rational point. In particular, we get the pullback map as  $G_k$ -hypercohomology of

$$\mathrm{id}_X^* \otimes^{\mathbb{L}} s^* : C^\bullet(X^s, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(X^s, \mathbb{Z}/\ell^n) \longrightarrow C^\bullet(X^s, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(EG_k, \mathbb{Z}/\ell^n) \simeq C^\bullet(X^s, \mu_{\ell^n}).$$

Let us first provide a lift of the pullback map  $\mathrm{id}_X^* \otimes^{\mathbb{L}} s^*$ , or more generally, of the analogue pullback map  $\mathrm{id}_U^* \otimes^{\mathbb{L}} s^* : C^\bullet(U \times_k X, \mu_{\ell^n}) \rightarrow C^\bullet(U, \mu_{\ell^n})$  for an open subscheme  $U \hookrightarrow X$ :

**4.11 Lemma.** *Let  $X/k$  be a geometrically connected smooth projective curve over a  $p$ -adic field  $k$  of genus  $\geq 2$ ,  $\mathfrak{X}/\mathfrak{o}$  a regular, proper, flat model s.t. the reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\mathrm{red}}/\mathbb{F}$  satisfies the assumptions of Thm. 2.4 (iii). Let  $\mathfrak{U} \hookrightarrow \mathfrak{X}$  be an open subscheme with generic fibre  $U \hookrightarrow X$ . Then for any  $\ell \neq p$  and any section  $s$  of  $\pi_1(X/k)$ , there is a canonical lift  $\mathrm{id}_{\mathfrak{U}}^* \otimes^{\mathbb{L}} \bar{s}_{\ell}^*$  of  $\mathrm{id}_U^* \otimes^{\mathbb{L}} s^*$  to  $\mathfrak{U} \times_{\mathfrak{o}} \mathfrak{X}$ :*

$$\begin{array}{ccc} C^\bullet(U \times_k X, \mu_{\ell^n}) & \xrightarrow{\mathrm{id}_U^* \otimes^{\mathbb{L}} s^*} & C^\bullet(U, \mu_{\ell^n}) \\ \uparrow \eta^* & & \uparrow \eta^* \\ C^\bullet(\mathfrak{U} \times_{\mathfrak{o}} \mathfrak{X}, \mu_{\ell^n}) & \xrightarrow{\mathrm{id}_{\mathfrak{U}}^* \otimes^{\mathbb{L}} \bar{s}_{\ell}^*} & C^\bullet(\mathfrak{U}, \mu_{\ell^n}). \end{array}$$

**Proof.** From  $\bar{s}_{\ell}^* : C^\bullet(\mathfrak{X}^{\mathrm{nr}}, \mathbb{Z}/\ell^n) \simeq C^\bullet(Y^s, \mathbb{Z}/\ell^n) \rightarrow C^\bullet(EG_{\mathbb{F}}, \mathbb{Z}/\ell^n)$  in  $\mathcal{D}^+(\underline{\mathrm{Mod}}_{G_{\mathbb{F}}})$  as in Cor. 4.9, we get

$$\mathrm{id}_{\mathfrak{U}}^* \otimes^{\mathbb{L}} \bar{s}_{\ell}^* : C^\bullet(\mathfrak{U}^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(\mathfrak{X}^{\mathrm{nr}}, \mathbb{Z}/\ell^n) \longrightarrow C^\bullet(\mathfrak{U}^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(EG_{\mathbb{F}}, \mathbb{Z}/\ell^n) \simeq C^\bullet(\mathfrak{U}^{\mathrm{nr}}, \mu_{\ell^n}).$$

Taking  $G_{\mathbb{F}}$ -hypercohomology, we get a canonical pullback map  $C^\bullet(\mathfrak{U} \times_{\mathfrak{o}} \mathfrak{X}, \mu_{\ell^n}) \rightarrow C^\bullet(\mathfrak{U}, \mu_{\ell^n})$ . Consider the commutative diagram

$$\begin{array}{ccc} C^\bullet(\mathfrak{U} \times_{\mathfrak{o}} \mathfrak{X}, \mu_{\ell^n}) & \xrightarrow{\mathrm{id}_{\mathfrak{U}}^* \otimes^{\mathbb{L}} \bar{s}_{\ell}^*} & C^\bullet(\mathfrak{U}, \mu_{\ell^n}) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma(G_{\mathbb{F}}, C^\bullet(U^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(X^{\mathrm{nr}}, \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{R}\Gamma(G_{\mathbb{F}}, C^\bullet(U^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(I_k, \mathbb{Z}/\ell^n)) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma(G_k, C^\bullet(U^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(X^{\mathrm{nr}}, \mathbb{Z}/\ell^n)) & \longrightarrow & \mathbb{R}\Gamma(G_k, C^\bullet(U^{\mathrm{nr}}, \mu_{\ell^n}) \otimes^{\mathbb{L}} C^\bullet(I_k, \mathbb{Z}/\ell^n)) \\ \downarrow & & \downarrow \\ C^\bullet(U \times_k X, \mu_{\ell^n}) & \xrightarrow{\mathrm{id}_U^* \otimes^{\mathbb{L}} s^*} & C^\bullet(U, \mu_{\ell^n}) : \end{array}$$

The upper square is induced on  $G_{\mathbb{F}}$ -hypercohomology by the  $\mathcal{D}^+(\underline{\mathrm{Mod}}_{G_{\mathbb{F}}})$ -square in Cor. 4.9 tensored with  $\eta^* : C^\bullet(\mathfrak{U}^{\mathrm{nr}}, \mu_{\ell^n}) \rightarrow C^\bullet(U^{\mathrm{nr}}, \mu_{\ell^n})$ . The middle square is induced by  $G_k \twoheadrightarrow G_{\mathbb{F}}$  (use that an object is flat in  $\underline{\mathrm{Mod}}_{\mathbb{Z}/\ell^n \mathbb{Z}[\Gamma]}$  if and only if it is flat as a  $\mathbb{Z}/\ell^n \mathbb{Z}$ -module). Finally, the lower square is induced on  $G_k$ -hypercohomology by the  $G_k$ -equivariant coverings  $U^s \rightarrow U^{\mathrm{nr}}$  resp.  $EG_k \rightarrow BG_k \times_{BG_{\mathbb{F}}} EG_{\mathbb{F}} \simeq BI_k$ . It follows that the compositions of the left resp. right vertical maps are the maps on cohomology induced by the respective immersions of the generic fibres.  $\square$

To get a canonical lift of the  $\ell$ -adic cycle class map  $\mathrm{cl}_s$ , we combine Gabber's Absolute Purity Theorem (see [Fuj02]) with Lem. 4.11 applied to the open subscheme given by the punctured model  $\mathfrak{X}^\bullet := \mathfrak{X} \setminus Y_{\mathrm{Sing}} \hookrightarrow \mathfrak{X}$ . Note that we can not work with the pullback map to  $\mathfrak{X}$  itself, since  $\Delta_{\mathfrak{X}}$  is a Cartier-divisor on  $\mathfrak{X} \times_{\mathfrak{o}} \mathfrak{X}$  if and only if  $\mathfrak{X}/\mathfrak{o}$  is smooth.

**4.12 Proposition.** *Let  $X/k$  be a geometrically connected smooth projective curve over a  $p$ -adic field  $k$  of genus  $\geq 2$ ,  $\mathfrak{X}/\mathfrak{o}$  a regular, proper, flat model s.t. the reduced special fibre  $Y = (\mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{F})_{\text{red}}/\mathbb{F}$  satisfies the assumptions of Thm. 2.4 (iii). Then for any  $\ell \neq p$  and any section  $s$  of  $\pi_1(X/k)$ , the induced  $\ell$ -adic cycle class  $\text{cl}_s$  admits a canonical lift  $\text{cl}_s^{\mathfrak{X}}$  to  $H^2(\mathfrak{X}, \mathbb{Z}_{\ell}(1))$ .*

**Proof.** First, note that  $\mathfrak{X}^{\bullet}$  still has generic fibre  $X$ . The diagonal  $\Delta_{\mathfrak{X}}$  of  $\mathfrak{X} \times_{\mathfrak{o}} \mathfrak{X}$  is Cartier outside of the collection of points  $(y, y)$  for  $y$  a singular point of  $Y$ . It follows, that  $\Delta_{\mathfrak{X}}|_{\mathfrak{X}^{\bullet} \times_{\mathfrak{o}} \mathfrak{X}}$  is a Cartier-divisor on  $\mathfrak{X}^{\bullet} \times_{\mathfrak{o}} \mathfrak{X}$ . By construction, its Chern-class  $\hat{c}_1[\mathfrak{X}^{\bullet} \times_{\mathfrak{o}} \mathfrak{X}]$  is a lift of the Chern class  $\hat{c}_1[\Delta_X]$  of the diagonal. It follows by Lem. 4.11, that  $(\text{id}_U^* \otimes^{\mathbb{L}} \bar{s}_{\ell}^*)(\hat{c}_1[\mathfrak{X}^{\bullet} \times_{\mathfrak{o}} \mathfrak{X}])$  is a canonical lift of  $\text{cl}_s = (\text{id}_U^* \otimes^{\mathbb{L}} s^*)(\hat{c}_1[\Delta_X])$  to  $H^2(\mathfrak{X}^{\bullet}, \mathbb{Z}_{\ell}(1))$ . But  $Y_{\text{Sing}} \hookrightarrow \mathfrak{X}$  is a closed immersion of regular schemes of pure codimension 2, so  $H^2(\mathfrak{X}, \mathbb{Z}_{\ell}(1)) \rightarrow H^2(\mathfrak{X}^{\bullet}, \mathbb{Z}_{\ell}(1))$  is an isomorphism by Gabber's Absolute Purity Theorem and we get a canonical lift  $\text{cl}_s^{\mathfrak{X}}$  of  $\text{cl}_s$  to  $H^2(\mathfrak{X}, \mathbb{Z}_{\ell}(1))$ .  $\square$

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